On totally magic cordial labeling

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(Received August 11, 2011; Revised July 19, 2013)

Abstract. A graph $G$ is said to have totally magic cordial (TMC) labeling with constant $C$ if there exists a mapping $f : V(G) \cup E(G) \rightarrow \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod{2}$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label $i$. In this paper, we investigate some new families of graphs that admit totally magic cordial labeling.

AMS 2010 Mathematics Subject Classification. 05C78.

Key words and phrases. Totally magic cordial, total sequential cordial, flower graph, ladder graph.

§1. Introduction

All graphs considered here are finite, simple and undirected. We follow the basic notations and terminologies of graph theory as in Harary [5]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. A detailed survey of graph labeling is available in [4]. The concept of cordial labeling was introduced by Cahit [1] and he proved that every tree is cordial, $K_n$ is cordial if $n \leq 3$, $K_{m,n}$ is cordial for all $m$ and $n$, the friendship graph $C_3^{(t)}$ is cordial if and only if $t \not\equiv 2 \pmod{4}$, all fans are cordial and the wheel graph $W_n$ is cordial if and only if $n \not\equiv 3 \pmod{4}$. In [2] he proved that a $k$-angular cactus with $t$ cycles is cordial if and only if $kt \not\equiv 2 \pmod{4}$. Further results on cordial labelings were discussed in [6, 7, 8, 9].

Based on cordial labeling Cahit [3] introduced another two well known graph labelings namely totally magic cordial labeling (TMC) and total sequential cordial labeling (TSC). In this paper, we show that the graph $G$ is TMC if and only if $G$ is TSC, a graph with number of vertices and number of edges differ by atmost 1 is TMC and also investigate that the TMC labelings of some families of graphs. In Theorem 10 [3], Cahit proved that the complete
A graph $K_n$ is TMC if and only if $n \in \{2, 3, 5, 6\}$. This observation is not correct. We rectify this error in Theorem 2.11.

We use the following definitions in the subsequent section:

**Definition 1.1.** A graph $G$ is said to have totally magic cordial (TMC) labeling with constant $C$ if there exists a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod{2}$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label $i$.

**Definition 1.2.** A graph $G$ is said to have total sequential cordial (TSC) labeling if there is a total mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that for each edge $e = \{a, b\}$, $f(e) = |f(a) - f(b)|$ and the condition $|n_f(0) - n_f(1)| \leq 1$ holds.

**Definition 1.3.** A wheel graph $W_n$ is obtained from a cycle $C_n$ by adding a new vertex and joining it to all the vertices of the cycle by an edge, then the new edges are called spokes of the wheel.

**Definition 1.4.** Flower graph $Fl_n(n \geq 3)$ is constructed from a wheel $W_n$ by attaching a pendant edge at each vertex of the $n$-cycle and by joining each pendant vertex to the central vertex.

**Definition 1.5.** Ladder graph $L_n(n \geq 2)$ is a product graph $P_2 \times P_n$ with $2n$ vertices and $3n - 2$ edges.

**Definition 1.6.** An $(n, t)$-kite graph is a cycle $C_n$ with a $t$-edge path (the tail) attached to one vertex.

**Definition 1.7.** An $n$-sun graph is a cycle $C_n$ with a pendant edge attached to each vertex of a cycle $C_n$.

**Definition 1.8.** A friendship graph $T_n(n \geq 2)$ is the one-point union of $t$ cycles of length $n$.

§2. Main Results

**Theorem 2.1.** If $G$ is a $(p, q)$ graph with $|p - q| \leq 1$ then $G$ is TMC.

*Proof.* If we assign 0 to all the edges of $G$ and 1 to all the vertices of $G$ then we get $C = 0$. If we assign 1 to all the edges of $G$ and 0 to all the vertices of $G$ then we get $C = 1$. In either case, $|n_f(0) - n_f(1)| = |p - q| \leq 1$. Clearly, $G$ is TMC.

**Corollary 2.2.** All trees, cycles ($n \geq 3$), unicyclic graphs, $(n, t)$-kite graphs ($n \geq 3$) and $n$-sun graphs ($n \geq 3$) are TMC.
Theorem 2.3. A graph $G$ is TMC if and only if $G$ is TSC.

Proof. A mapping $f : V(G) \cup E(G) \to \{0, 1\}$ is a TMC labeling with constant 0 if and only if $f$ is a TSC labeling, and $f$ is a TMC labeling with constant 1 if and only if $\tilde{f}$ is a TSC labeling, where $\tilde{f}$ is defined by $\tilde{f}(x) = 1 - f(x)$, for all $x \in V(G) \cup E(G)$. Hence a graph $G$ has a TMC labeling if and only if $G$ has a TSC labeling. \qed

Cahit [3] proved that every cordial graph is TSC and the friendship graph $T_n$ is TMC for all $n \geq 2$. Hence, we obtain the following results:

Corollary 2.4. Every cordial graph is TMC.

Corollary 2.5. The friendship graph $T_n$ is TMC for all $n \geq 2$.

Lemma 2.6. The flower graph $F_l_n$ is TMC for $n \geq 3$.

Proof. Let $V = \{u, u_i, v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{u_i v_i, u_i u_{i+1}, u_{i+1} v_i | 1 \leq i \leq n\} \cup \{u_n u_1\}$ be the edge set for $n \geq 3$. Clearly, $|V| = 2n + 1$ and $|E| = 4n$. Define $f : V \cup E \to \{0, 1\}$ as follows: $f(u) = 0$, $f(u_i) = 0$, $f(v_i) = 1$, $f(u_i u_i) = 1$, $f(u_{i+1} v_i) = 0$ for $0 \leq i \leq n$ and $f(u_{i+1} u_{i+1}) = f(u_n u_1) = 1$ for $0 \leq i < n$. Clearly, $f(a) + f(b) + f(ab) \equiv 1 \pmod{2}$ for all $ab \in E$. Also, $n_f(0) = n_f(1) = 3n + 1$. Thus, $|n_f(0) - n_f(1)| \leq 1$. Hence, $F_l_n$ is TMC for $n \geq 3$. \qed

Lemma 2.7. The ladder graph $L_n$ is TMC for all $n \geq 2$.

Proof. Let the vertex set be $V = \{u_i, v_i | 1 \leq i \leq n\}$ and the edge set be $E = \{u_i v_i | 1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1} | 1 \leq i < n\}$. Clearly, $|V| = 2n$ and $|E| = 3n - 2$. Define $f : V \cup E \to \{0, 1\}$ as follows: $f(u_i) = 0$ for $i = 1, 2, \ldots, n$ and $f(u_{i+1} u_{i+1}) = 1$ for $i = 1, 2, \ldots, n - 1$. $f(v_i) = f(v_{i+1}) = 0$, $f(u_i v_i) = f(u_{i+1} v_{i+1}) = 1$ for $i \equiv 1 \pmod{4}$, $f(v_i) = f(v_{i+1}) = 1$, $f(u_i v_i) = f(u_{i+1} v_{i+1}) = 0$ for $i \equiv 3 \pmod{4}$ and $f(v_i v_{i+1}) = \begin{cases} 1 & \text{if } i \text{ is odd}, \\ 0 & \text{if } i \text{ is even.} \end{cases}$

Clearly, $C = 1$ and $n_f(0) = n_f(1) + 1 = \frac{5n-1}{2}$ if $n$ is odd and $n_f(0) = n_f(1) = \frac{5n-2}{2}$ if $n$ is even. Hence, the ladder graph $L_n$ is TMC for all $n \geq 2$. \qed

Lemma 2.8. If $G$ is a graph obtained by identifying a vertex of the cycle $C_m (m \geq 3)$ with each vertex of the cycle $C_n (n \geq 3)$ then $G$ is TMC.

Proof. Let $V(G) = \{u_i | 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{u_i u_{i+1} | 1 \leq i \leq m - 1, 1 \leq j \leq n\} \cup \{u_m u_1 | 1 \leq j \leq n\}$
the complete graph can be decomposed as 
\( K_n \). Clearly, \(|V(G)| = mn \) and \(|E(G)| = mn + n \). Define \( f : V(G) \cup E(G) \to \{0, 1\} \) as follows: For \( j = 1, 2, \ldots, n \),
\[
f(u^j_2) = \begin{cases} 
0 & \text{if } j \text{ is odd}, \\
1 & \text{if } j \text{ is even}
\end{cases}
\]
and \( f(u^i_2) = 0 \) for \( i \neq 2 \) and \( i = 1, 3, \ldots, m \).
\[
f(u^i_1 u^j_2) = f(u^j_2 u^i_3) = \begin{cases} 
1 & \text{if } j \text{ is odd,} \\
0 & \text{if } j \text{ is even.}
\end{cases}
\]
For \( i = 3, 4, \ldots, m \), \( f(u^i_1 u^{i+1}_2) = 1 \) and for \( j = 1, 2, \ldots, n - 1 \), \( f(u^i_1 u^{i+1}_1) = f(u^i_1 u^1_1) = 1 \). Clearly, \( C = 1 \) and
\[
n_f(1) = \begin{cases} 
n_f(0) + 1 & \text{if } j \text{ is odd,} \\
n_f(0) & \text{if } j \text{ is even.}
\end{cases}
\]
Hence, \( G \) is TMC.
\[\square\]

**Theorem 2.9.** If \( G_1(p_1, q_1) \) and \( G_2(p_2, q_2) \) are two disjoint TMC graphs and \( p_1 = q_1 \) or \( p_2 = q_2 \) then \( G_1 \cup G_2 \) is also TMC.

**Proof.** Let \( f \) and \( g \) be TMC labeling of \( G_1 \) and \( G_2 \) respectively with the same constant \( C \). Without loss of generality, we assume that \( p_1 = q_1 \). Then \( n_f(0) = n_g(0) \). Define \( h : V(G_1 \cup G_2) \cup E(G_1 \cup G_2) \to \{0, 1\} \) by \( h/V(G_1) \cup E(G_1) = f \) and \( h/V(G_2) \cup E(G_2) = g \). Now \( n_h(0) = n_f(0) + n_g(0) = n_f(1) \) if \( n_g(0) = n_g(1) \). Similarly, \( n_h(0) = n_h(1) + 1 \) if \( n_g(0) = n_g(1) + 1 \) and \( n_h(1) = n_h(0) + 1 \) if \( n_g(1) = n_g(0) + 1 \). Thus, \( h \) is a TMC labeling of \( G_1 \cup G_2 \) and hence, \( G_1 \cup G_2 \) is TMC.
\[\square\]

**Corollary 2.10.** The disjoint union of cycle with the TMC graph \( G \) is TMC.

**Theorem 2.11.** The complete graph \( K_n \) is TMC if and only if
\[
\begin{cases} 
\sqrt{4k + 1} & \text{has an integer value when } n = 4k, \\
\sqrt{k + 1} \text{ or } \sqrt{k} & \text{has an integer value when } n = 4k + 1, \\
\sqrt{4k + 5} \text{ or } \sqrt{4k + 1} & \text{has an integer value when } n = 4k + 2, \\
\sqrt{k + 1} & \text{has an integer value when } n = 4k + 3.
\end{cases}
\]

**Proof.** Assume that \( f \) is a TMC labeling of \( K_n \). Without loss of generality, we assume that \( C = 1 \). Then for any edge \( e = uv \in E(K_n) \), we have either \( f(e) = f(u) = f(v) = 1 \) or \( f(e) = f(u) = 0 \) and \( f(v) = 1 \) or \( f(e) = f(v) = 0 \) and \( f(u) = 1 \) or \( f(u) = f(v) = 0 \) and \( f(e) = 1 \). Hence, under the labeling \( f \), the complete graph can be decomposed as \( K_n = K_p \cup K_r \cup K_{p,r} \), where \( K_p \) is the subgraph whose vertices and edges are labeled with 1, \( K_r \) is the sub
graph whose vertices labeled with 0 and its edges labeled with 1 and $K_{p,r}$ is the subgraph of $K_n$ with the bipartition $V(K_p) \cup V(K_r)$ in which the edges are labeled with 0. Thus, we have $n_f(0) = r + pr$ and $n_f(1) = p + \frac{p(p-1)}{2} + \frac{r(r-1)}{2}$.

Also, for any TMC labeling $f$ of $K_n$ we must have the following:

(i) $n_f(0) = n_f(1)$ if $n \equiv 0, 3 \pmod{4}$.

(ii) $n_f(1) = n_f(0) + 1$ or $n_f(0) = n_f(1) + 1$ if $n \equiv 1, 2 \pmod{4}$.

Case i. $n \equiv 0, 3 \pmod{4}, n > 2$.

Then $n_f(0) = n_f(1)$, which implies $p^2 + p(1-2r) + r^2 - 3r = 0$. Since $p = n-r$, we have $4r^2 - 4r(n+1) + n^2 + n = 0$. Hence, $r = \frac{1}{2} \left[ (n+1) \pm \sqrt{n+3} \right]$. Since $r$ is the order of subgraph $K_r$, it can be seen that $K_{4k}$, $k \geq 1$, is TMC only if $\sqrt{4k+1}$ has an integer value and $K_{4k+3}$, $k \geq 0$, is TMC only if $\sqrt{k+1}$ has an integer value.

Case ii. $n \equiv 1, 2 \pmod{4}, n > 2$.

Then, $n_f(1) = n_f(0) + 1$ or $n_f(0) = n_f(1) + 1$.

If $n_f(1) = n_f(0) + 1$, $p^2 + p(1-2r) + r^2 - 3r + 2 = 0$. Since $p = n-r$, $4r^2 - 4r(n+1) + n^2 + n - 2 = 0$. Hence, $r = \frac{1}{2} \left[ (n+1) \pm \sqrt{n+3} \right]$. For $k \geq 1$, $K_{4k+1}$ is TMC only if $\sqrt{k+1}$ has an integer value and for $k \geq 1$, $K_{4k+2}$ is TMC only if $\sqrt{4k+5}$ has an integer value.

Again, if $n_f(0) = n_f(1) + 1$, $p^2 + p(1-2r) + r^2 - 3r + 2 = 0$. Since $p = n-r$, $4r^2 - 4r(n+1) + n^2 + n + 2 = 0$. Hence, $r = \frac{1}{2} \left[ (n+1) \pm \sqrt{n-1} \right]$. For $k \geq 1$, $K_{4k+1}$ is TMC only if $\sqrt{k}$ has an integer value and for $k \geq 1$, $K_{4k+2}$ is TMC only if $\sqrt{4k+1}$ has an integer value.

Thus, the complete graph $K_n$ is TMC if and only if

$$\begin{cases} 
\sqrt{4k+1} \text{ has an integer value when } n = 4k, \\
\sqrt{k+1} \text{ or } \sqrt{k} \text{ has an integer value when } n = 4k+1, \\
\sqrt{4k+5} \text{ or } \sqrt{4k+1} \text{ has an integer value when } n = 4k+2, \\
\sqrt{k+1} \text{ has an integer value when } n = 4k+3. 
\end{cases}$$

Acknowledgement: The authors are thankful to the referee for the valuable comments for a better presentation of the paper.

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