Totally vertex-magic cordial labeling

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(Received September 21, 2010; Revised May 26, 2013)

Abstract. In this paper, we introduce a new labeling called Totally Vertex-
Magic Cordial (TVMC) labeling. A graph $G(p, q)$ is said to be TVMC with a
constant $C$ if there is a mapping $f : V(G) \cup E(G) \rightarrow \{0, 1\}$ such that

$$\left[ f(a) + \sum_{b \in N(a)} f(ab) \right] \equiv C \pmod{2}$$

for all vertices $a \in V(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $N(a)$ is the set of
vertices adjacent to the vertex $a$ and $n_f(i)(i = 0, 1)$ is the sum of the number
of vertices and edges with label $i$.

AMS 2010 Mathematics Subject Classification. 05C78.

Key words and phrases. Totally vertex-magic cordial, sun graph, friendship
graph.

§1. Introduction

All graphs considered here are finite, simple and undirected. The set of vertices
and edges of a graph $G$ will be denoted by $V(G)$ and $E(G)$ respectively, and
let $p = |V(G)|$ and $q = |E(G)|$. A labeling of a graph $G$ is a mapping that
carries a set of graph elements usually the vertices and/or edges, into a set
of numbers, usually integers, called labels. Many kinds of labelings have been
studied and an excellent survey of graph labeling can be found in Gallian [3].
For all other terminology and notation we follow Harary [4]. The concept
of cordial labeling was introduced by Cahit [1]. A binary vertex labeling
$f : V(G) \rightarrow \{0, 1\}$ induces an edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined by

$$f^*(uv) = |f(u) - f(v)|.$$ 

Such a labeling is called cordial if the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ are satisfied, where $v_f(i)$ and
e$_f(i)(i = 0, 1)$ are the number of vertices and edges with label $i$ respectively.
A graph is called cordial if it admits cordial labeling.
Totally Magic Cordial (TMC) labeling was introduced by Cahit in [2] as a modification of total edge-magic labeling. A \((p, q)\) graph \(G\) is said to have a totally magic cordial labeling with constant \(C\) if there exists a mapping \(f : V(G) \cup E(G) \rightarrow \{0, 1\}\) such that \(f(a) + f(b) + f(ab) \equiv C \pmod{2}\) for all edges \(ab \in E(G)\) provided the condition \(|f(0) - f(1)| \leq 1\), where \(f(0) = v_f(0) + e_f(0)\), \(f(1) = v_f(1) + e_f(1)\) and \(v_f(i)\), \(e_f(i)\) \((i = 0, 1)\) are the number of vertices and edges with label \(i\), respectively. It is proved that the graphs \(K_{m,n}\) \((m, n > 1)\), trees and \(K_n\) for \(n = 2, 3, 5\) or 6 have TMC labeling.

J. A. MacDougall et al. introduced the concept of vertex-magic total labeling in [6]. A one-to-one map \(\lambda\) from \(V \cup E\) onto the integers \(\{1, 2, ..., p + q\}\) is a vertex-magic total labeling if there is a constant \(k\) so that for every vertex \(x\), \(\lambda(x) + \sum \lambda(xy) = k\), where the sum is over all vertices \(y\) adjacent to \(x\). The sum \(\lambda(x) + \sum \lambda(xy)\) is called the weight of the vertex \(x\) and is denoted by \(wt(x)\). The constant \(k\) is called the magic constant for \(\lambda\). In this paper, we modify the vertex-magic total labeling into a new labeling called totally vertex magic cordial labeling and we examine the totally vertex magic cordiality of some graphs.

\section{Totally vertex-magic cordial labeling}

In this section, we define totally vertex-magic cordial labeling and we prove vertex-magic total graph is totally vertex-magic cordial.

\begin{definition}
A \((p, q)\) graph \(G\) is said to have a totally vertex-magic cordial (TVMC) labeling with constant \(C\) if there is a mapping \(f : V(G) \cup E(G) \rightarrow \{0, 1\}\) such that
\[
\left[ f(a) + \sum_{b \in N(a)} f(ab) \right] \equiv C \pmod{2}
\]
for all vertices \(a \in V(G)\) provided the condition, \(|n_f(0) - n_f(1)| \leq 1\) is held, where \(N(a)\) is the set of vertices adjacent to a vertex \(a\) and \(n_f(i)\) \((i = 0, 1)\) is the sum of the number of vertices and edges with label \(i\).
\end{definition}

A graph is called totally vertex-magic cordial if it admits totally vertex-magic cordial labeling.

\begin{theorem}
If \(G\) is a vertex-magic total graph then \(G\) is totally vertex-magic cordial.
\end{theorem}

\begin{proof}
Let \(f\) be a vertex-magic total labeling of a graph \(G\) with \(p\) vertices and \(q\) edges and with weight \(k\). Define \(g : V(G) \cup E(G) \rightarrow \{0, 1\}\) by \(g(v) \equiv f(v)\)
(mod 2) if $v \in V(G)$ and $g(e) \equiv f(e) \pmod{2}$ if $e \in E(G)$. Then, $C = 0$ if $k$ is even and $C = 1$ if $k$ is odd. Since there are exactly $\left\lceil \frac{p+q}{2} \right\rceil$ odd integers and $\left\lfloor \frac{p+q}{2} \right\rfloor$ even integers in the set $\{1, 2, 3, ..., p+q\}$ we have, $|n_f(0) - n_f(1)| \leq 1$. Hence, $g$ is a totally vertex-magic cordial labeling of $G$.

§3.

Totally vertex-magic cordial labeling of a complete graph $K_n$

H. K. Krishnappa et al. [5] proved that $K_n(n \geq 1)$ admits vertex-magic total labeling. In this section, we use another technique to prove $K_n(n \geq 1)$ is totally vertex-magic cordial. Let $V = \{v_i|1 \leq i \leq n\}$ be the vertex set and $E = \{v_iv_j|i \neq j, 1 \leq i, j \leq n\}$ be the edge set of $K_n$. We use the following symmetric matrix to label the vertices and the edges of $K_n$, which is called the label matrix for $K_n$.

$$
\begin{bmatrix}
  e_{11} & e_{21} & e_{31} & e_{41} & e_{51} & \cdots & e_{n1} \\
  e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & \cdots & e_{n2} \\
  e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & \cdots & e_{n3} \\
  e_{41} & e_{42} & e_{43} & e_{44} & e_{45} & \cdots & e_{n4} \\
  e_{51} & e_{52} & e_{53} & e_{54} & e_{55} & \cdots & e_{n5} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  e_{n1} & e_{n2} & e_{n3} & e_{n4} & e_{n5} & \cdots & e_{nn}
\end{bmatrix}
$$

The entries in the main diagonal represent the vertex labels, $f(v_i) = e_{ii}$ and the other entries $e_{ij}$, $i \neq j$ represent the edge labels, $f(v_iv_j) = e_{ij}$. Thus the weight of a vertex $v_i$ is the sum of the elements either in the $i$th row or in the $i$th column.

**Theorem 3.1.** The complete graph $K_n$ is TVMC for all $n \geq 1$.

**Proof.** Let $K_n$ be the complete graph with $n$ vertices. We consider the following three cases:

**Case i.** $n \equiv 0 \pmod{4}$.

We construct the label matrix for $K_n$ as follows:

$$
e_{ij} = \begin{cases} 
0 & \text{when } i + j \equiv 0, 1 \pmod{4}, \\
1 & \text{when } i + j \equiv 2, 3 \pmod{4}.
\end{cases}
$$

Then for each vertex $v_r$, $1 \leq r \leq n$, the weight $\text{wt}(v_r)$ is the sum of the elements in the $r$th row or in the $r$th column. Hence,

$$
\text{wt}(v_r) = \sum_{j=1}^{r} e_{rj} + \sum_{i=r+1}^{n} e_{ir} = \frac{n}{2} \equiv 0 \pmod{2}.
$$
Also \( n_f(0) = n_f(1) = \frac{n^2+n}{4} \). Therefore, \(|n_f(0) - n_f(1)| = 0\).

**Case ii.** \( n \equiv 2 \pmod{4} \).

We construct the label matrix as follows: when \( j \equiv 0, 1 \pmod{4} \),

\[
e_{ij} = \begin{cases} 
1 & \text{if } i \text{ is odd}, \\
0 & \text{if } i \text{ is even}
\end{cases}
\]

and when \( j \equiv 2, 3 \pmod{4} \),

\[
e_{ij} = \begin{cases} 
0 & \text{if } i \text{ is odd}, \\
1 & \text{if } i \text{ is even}.
\end{cases}
\]

Then

\[
\text{wt}(v_r) = \sum_{j=1}^{r} e_{rj} + \sum_{i=r+1}^{n} e_{ir} = n \frac{r}{2} \equiv 1 \pmod{2}.
\]

Also \( n_f(0) = \frac{n^2+n-2}{4} \) and \( n_f(1) = \frac{n^2+n+2}{4} \). Hence, \(|n_f(0) - n_f(1)| = 1\).

**Case iii.** \( n \) is odd.

We construct the label matrix as follows: when \( i + j \leq n \),

\[
e_{ij} = \begin{cases} 
1 & \text{if } i \text{ is odd}, \\
0 & \text{if } i \text{ is even}
\end{cases}
\]

and when \( i + j > n \),

\[
e_{ij} = \begin{cases} 
1 & \text{if } j \text{ is odd}, \\
0 & \text{if } j \text{ is even}.
\end{cases}
\]

We have

\[
\text{wt}(v_r) = \sum_{j=1}^{r} e_{rj} + \sum_{i=r+1}^{n-r} e_{ir} + \sum_{i=n-r+1}^{n} e_{ir} \text{ if } 1 \leq r < \frac{n+1}{2};
\]

\[
\text{wt}(v_r) = \sum_{j=1}^{r-1} e_{rj} + \sum_{i=r}^{n} e_{ir} \text{ if } r = \frac{n+1}{2};
\]

\[
\text{wt}(v_r) = \sum_{j=1}^{n-r} e_{rj} + \sum_{j=n-r+1}^{r-1} e_{rj} + \sum_{i=r}^{n} e_{ir} \text{ if } \frac{n+1}{2} < r < n;
\]

and \( \text{wt}(v_r) = \sum_{j=1}^{n} e_{rj} \) if \( r = n \).

The weights of the vertices for \( n = 4k + 1 \) and \( n = 4k + 3 \) are summarized in the following tables:

When \( n = 4k + 1 \),
\begin{array}{|c|c|c|c|c|}
\hline
 & 1 \leq r < \frac{n+1}{2} & r = \frac{n+1}{2} & \frac{n+1}{2} < r < n & r = n \\
\hline
r \text{ is odd} & 2k + r & n \times (r \mod 2) & 6k - r + 2 & \equiv 1 \mod 2 \\
 & \equiv 1 \mod 2 & \equiv 1 \mod 2 & \equiv 1 \mod 2 & \equiv 1 \mod 2 \\
\hline
r \text{ is even} & 2k - r + 1 & - & r - 2k - 1 & - \\
 & \equiv 1 \mod 2 & - & \equiv 1 \mod 2 & - \\
\hline
\end{array}

When \( n = 4k + 3 \),

\begin{array}{|c|c|c|c|c|}
\hline
 & 1 \leq r < \frac{n+1}{2} & r = \frac{n+1}{2} & \frac{n+1}{2} < r < n & r = n \\
\hline
r \text{ is odd} & 2k + r + 1 & - & 6k - r + 5 & \equiv 0 \mod 2 \\
 & \equiv 0 \mod 2 & - & \equiv 0 \mod 2 & \equiv 0 \mod 2 \\
\hline
r \text{ is even} & 2k - r + 2 & n \times (r \mod 2) & r - 2k - 2 & - \\
 & \equiv 0 \mod 2 & \equiv 0 \mod 2 & \equiv 0 \mod 2 & - \\
\hline
\end{array}

Also if \( n = 4k + 1 \), then \( n_f(0) = \frac{n^2 + n - 2}{4} \), \( n_f(1) = \frac{n^2 + n + 2}{4} \); if \( n = 4k + 3 \),
then \( n_f(0) = n_f(1) = \frac{n^2 + n}{4} \) and hence, \( |n_f(0) - n_f(1)| \leq 1 \). Therefore, \( K_n \) is TVMC for all \( n \geq 1 \).

\section{Totally vertex-magic cordial labeling of a complete bipartite graph \( K_{m,n} \)}

J. A. MacDougall et al. \cite{6} proved that there is a vertex-magic total labeling for a complete bipartite graph \( K_{m,m} \) for all \( m > 1 \). Also they conjectured that there is a vertex-magic total labeling for a complete bipartite graph \( K_{m,m+1} \).

In this section, we prove the bipartite graph \( K_{m,n} \) admits TVMC labeling whenever \( |m - n| \leq 1 \). We consider the complete bipartite graph \( K_{m,n} \) with the vertex set \( \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \) and the edge set \( \{e_{ij} = u_iv_j|1 \leq i \leq m, 1 \leq j \leq n\} \). We use the following \((m + 1) \times (n + 1)\) matrix to label the vertices and the edges of \( K_{m,n} \):

\[
\begin{bmatrix}
- & c_{01} & c_{02} & \cdots & c_{0n} \\
- & - & - & - & - \\
\vdots & c_{10} & c_{11} & c_{12} & \cdots & c_{1n} \\
\vdots & c_{20} & c_{21} & c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
c_{m0} & c_{m1} & c_{m2} & \cdots & c_{mn} \\
\end{bmatrix}
\]

The entries in the first row \( c_{i0}(1 \leq i \leq m) \) represent the labels of the vertices \( u_i(1 \leq i \leq m) \), the entries in the first column \( c_{0j}(1 \leq j \leq n) \) represent the labels of the vertices \( v_j(1 \leq j \leq n) \) and the other entries \( c_{ij} \) represent the labels of the edges \( u_iv_j(1 \leq i \leq m, 1 \leq j \leq n) \). That is, \( f(u_i) = c_{i0} \), \( f(v_j) = c_{0j} \) and \( f(u_iv_j) = c_{ij} \) for \( 1 \leq i \leq m, 1 \leq j \leq n \).
Lemma 4.1. $K_{m,m+1}$ is TVMC for all $m \geq 1$.

Proof. Define

$$c_{ij} = \begin{cases} 
1 & \text{if } i = 0 \text{ or } j = 0 \text{ and } i + j \text{ is odd,} \\
0 & \text{if } i = 0 \text{ or } j = 0 \text{ and } i + j \text{ is even,} \\
1 & \text{if } i \neq 0, j \neq 0 \text{ and } i + j \leq m + 1, \\
0 & \text{if } i \neq 0, j \neq 0 \text{ and } i + j > m + 1.
\end{cases}$$

Then $n_f(0) = \frac{m^2 + 3m}{2}$, $n_f(1) = \frac{m^2 + 3m + 2}{2}$ and hence, $|n_f(0) - n_f(1)| = 1$. The weights of vertices $u_i$ and $v_j$ are summarized in the following table:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$i$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ is even</td>
<td>$m + 1 - i$</td>
<td>$m + 2 - i$</td>
<td>$m + 1 - j$</td>
</tr>
<tr>
<td>$m \equiv 1 \pmod{2}$</td>
<td>$m \equiv 1 \pmod{2}$</td>
<td>$m \equiv 1 \pmod{2}$</td>
<td>$m \equiv 1 \pmod{2}$</td>
</tr>
<tr>
<td>$m$ is odd</td>
<td>$m + 1 - i$</td>
<td>$m + 2 - i$</td>
<td>$m + 1 - j$</td>
</tr>
<tr>
<td>$m \equiv 0 \pmod{2}$</td>
<td>$m \equiv 0 \pmod{2}$</td>
<td>$m \equiv 0 \pmod{2}$</td>
<td>$m \equiv 0 \pmod{2}$</td>
</tr>
</tbody>
</table>

Therefore, $K_{m,m+1}$ is TVMC for all $m \geq 1$.

Lemma 4.2. $K_{m,m}$ is TVMC if $m$ is odd.

Proof. Define

$$c_{ij} = \begin{cases} 
1 & \text{if } i + j \text{ is odd,} \\
0 & \text{if } i + j \text{ is even.}
\end{cases}$$

Then $n_f(0) = \frac{m^2 + 2m - 1}{2}$, $n_f(1) = \frac{m^2 + 2m + 1}{2}$ and hence, $|n_f(0) - n_f(1)| = 1$. The weight of each vertex is

$$\frac{m + 1}{2} \equiv \begin{cases} 
1 & \text{if } m \equiv 1 \pmod{4}, \\
0 & \text{if } m \equiv 3 \pmod{4}.
\end{cases}$$

Therefore, $K_{m,m}$ is TVMC for odd values of $m$.

Lemma 4.3. $K_{m,m}$ is TVMC if $m \equiv 0 \pmod{4}$.

Proof. Let $m = 4k$. Define $c_{i0} = 0$, $c_{0j} = 0$ and for $i \neq 0$ and $j \neq 0$,

$$c_{ij} = \begin{cases} 
1 & \text{if } |i - j| = 0, 1, 2, \ldots, \frac{m}{4} \text{ and } \frac{3m}{4}, \ldots, m - 1, \\
0 & \text{otherwise.}
\end{cases}$$

Then, $\text{wt}(v_j) = \text{wt}(u_i) = \frac{m^2}{2} + 1 = 2k + 1 \equiv 1 \pmod{2}$ for all $i$ and $j$. Also $n_f(0) = n_f(1) = \frac{m^2 + 2m}{2}$. Thus, $|n_f(0) - n_f(1)| = 0$. Hence, $K_{m,m}$ is TVMC for $m \equiv 0 \pmod{4}$.
Lemma 4.4. $K_{m,m}$ is TVMC if $m \equiv 2 \pmod{4}$.

Proof. Let $m = 4k + 2$. Define $c_{i0} = 0$, $c_{0j} = 1$ and for $i \neq 0$ and $j \neq 0$,

$$c_{ij} = \begin{cases} 1 & \text{if } j \text{ is odd;} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

Then, $wt(v_j) = m + 1 \equiv 1 \pmod{2}$ if $j$ is odd, $wt(v_j) = 1$ if $j$ is even and $wt(u_i) = \frac{m}{2} \equiv 1 \pmod{2}$. Also $n_f(0) = n_f(1) = \frac{m^2 + 2m}{2}$ and hence, $|n_f(0) - n_f(1)| = 0$. Thus, $K_{m,m}$ is TVMC for $m \equiv 2 \pmod{4}$. \hfill \Box

Lemma 4.5. $K_{m,n}$ is TVMC if $|m - n| \leq 1$.

Proof. The proof follows from Lemmas 4.1, 4.2, 4.3 and 4.4. \hfill \Box

§5. Totally vertex-magic cordial (TVMC) labelings of some graphs

J. A. MacDougall et al. [6] proved that not all trees have a vertex-magic total labeling. Also J. A. MacDougall et al. [7] proved that the friendship graph $T_n$ has no vertex-magic total labeling for $n > 3$. In the subsequent theorems we prove all trees are TVMC, the friendship graph $T_n$ for $n \geq 1$ is TVMC and also we examine the totally vertex magic cordiality of flower graph, $P_n + P_2$ and $P_n + K_2$.

Theorem 5.1. If $G$ is a $(p, q)$ graph with $|p - q| \leq 1$, then $G$ is TVMC with $C = 1$.

Proof. Assign 0 to all the edges and 1 to all the vertices of $G$. Then weight of each vertex is 1 and $|n_f(0) - n_f(1)| = |p - q| \leq 1$. Hence, $G$ is TVMC. \hfill \Box

Corollary 5.2. All cycles $(n \geq 3)$, trees and unicycle graphs are TVMC with $C = 1$.

A flower graph $F_{l_n}$ is constructed from a wheel $W_n$ by attaching a pendant edge at each vertex of the $n$-cycle and by joining each pendant vertex to the central vertex. We prove that $F_{l_n}$ admits TVMC labeling.

Theorem 5.3. The flower graph $F_{l_n}$ for $n \geq 3$ is TVMC with $C = 0$.

Proof. Let $V = \{u, u_i, v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{uu_i, u_i v_i, vv_i | 1 \leq i \leq n\} \cup \{u_j u_{j+1} | 1 \leq j \leq n - 1\} \cup \{u_n u_1\}$ be the edge set for $n \geq 3$. Clearly, $|V| = 2n + 1$ and $|E| = 4n$. Define $f : V \cup E \to \{0, 1\}$ as follows: For $1 \leq i \leq n$, $f(u_i) = 1$, $f(v_i) = 0$, $f(uu_i) = 1$, $f(u_i v_i) = 0$, $f(vv_i) = 0$ and for $1 \leq j \leq n - 1$, $f(u_j u_{j+1}) = 1$, $f(u_n u_1) = 1$ and

$$f(u) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$
We prove that the weight of each vertex is constant modulo 2.

\[
\text{wt}(u) = f(u) + \sum_{i=1}^{n} f(uv_i) + \sum_{i=1}^{n} f(uu_i) = \begin{cases} 
n & \text{if } n \text{ is even,} 
n +1 & \text{if } n \text{ is odd.}
\end{cases}
\]

Hence, \( \text{wt}(u) \equiv 0 \pmod{2} \). Further, for \( 1 \leq i \leq n \), \( \text{wt}(u_i) = 4 \equiv 0 \pmod{2} \) and \( \text{wt}(v_i) = 0 \). Also \( |n_f(0) - n_f(1)| \leq 1 \). Therefore, \( F_l_n \) is TVMC for \( n \geq 3 \).

The friendship graph \( T_n(n \geq 1) \) consists of \( n \) triangles with a common vertex.

**Theorem 5.4.** The friendship graph \( T_n \) for \( n \geq 1 \) is TVMC with \( C = 0 \).

**Proof.** Let \( V = \{u, u_i, v_i|1 \leq i \leq n\} \) and \( E = \{uv_i, u_iv_i, uu_i|1 \leq i \leq n\} \) be the vertex set and the edge set, respectively. Define \( f : V \cup E \to \{0, 1\} \) as follows:

\[
f(u_i) = 0, \quad f(v_i) = 1 \quad \text{and} \quad f(u) = \begin{cases} 
0 & \text{if } n \text{ is even,} 
n +1 & \text{if } n \text{ is odd.}
\end{cases}
\]

\( f(uu_i) = 0, \quad f(u_iv_i) = 0, \quad f(v_iu) = 1, \quad \text{and for} \quad \left\lfloor \frac{n}{2} \right\rfloor < i \leq n, \quad f(uu_i) = 1, \quad f(u_iv_i) = 1 \quad \text{and} \quad f(v_iu) = 0. \)

It can easily be verified that \( \text{wt}(u_i) \equiv \text{wt}(v_i) \equiv 0 (\pmod{2}) \). Also \( n_f(0) = \left\lfloor \frac{5n+1}{2} \right\rfloor \) and \( n_f(1) = \left\lfloor \frac{5n+1}{2} \right\rfloor \). Hence, \( |n_f(0) - n_f(1)| \leq 1 \). Therefore, \( T_n \) for \( n \geq 1 \) is TVMC with \( C = 0 \).

Let \( G \) and \( H \) be any two graphs. Let \( u \) be any vertex of \( G \) and \( v \) be any vertex of \( H \). Then \( G \@ H \) is a graph obtained by identifying the vertices \( u \) and \( v \).

**Theorem 5.5.** If \( G \) is TVMC with \( C = 1 \), then \( G \@ T \) is also TVMC with \( C = 1 \) for any tree \( T \).

**Proof.** Let \( f \) be the TVMC labeling of \( G \) with \( C = 1 \). Assign 0 to all the edges and 1 to all the vertices of \( T \). Identify a vertex \( u \in V(G) \) with a vertex \( v \in V(T) \) and take this new vertex as \( w \). Define a labeling \( g \) for \( G \@ T \) as follows:

\[
g(a) = \begin{cases} 
f(a) & \text{if } a \in V(G), 
1 & \text{if } a \in V(T) \text{ and } a \neq w,
\end{cases}
\]

and

\[
g(e) = \begin{cases} 
f(e) & \text{if } e \in E(G), 
0 & \text{if } e \in E(T).
\end{cases}
\]
Then the weight of the identified vertex \( w \) is,

\[
\text{wt}_{G@T}(w) = g(w) + \sum_{x \in N(w)} g(xw) = f(u) + \sum_{x \in N(u)} f(xu) + \sum_{y \in N(w)} f(yu) = f(u) + \sum_{x \in N(u)} f(xu)
\]

\[
= \text{wt}(u) \equiv 1 \pmod{2}.
\]

For each \( a \in V(G@T) \) with \( a \neq w \), \( \text{wt}_{G@T}(a) = \text{wt}_{G}(a) \equiv 1 \pmod{2} \) if \( a \in V(G) \) and \( \text{wt}_{G@T}(a) = 1 \) if \( a \in V(T) \). Also \( |n_{g}(0) - n_{g}(1)| = |n_{f}(0) - n_{f}(1)| \leq 1 \). Hence, \( G@T \) is also TVMC with \( C = 1 \).

The join of two graphs \( G_{1} \) and \( G_{2} \) is denoted by \( G_{1} + G_{2} \) and it consists of \( G_{1} \cup G_{2} \) and all the lines joining \( V(G_{1}) \) with \( V(G_{2}) \).

**Theorem 5.6.** \( P_{n} + P_{2} \) is TVMC for \( n \geq 1 \).

**Proof.** Let \( G = P_{n} + P_{2} \). We denote the vertices of \( P_{n} \) in \( G \) by \( u_{1}, u_{2}, \ldots, u_{n} \) and the vertices of \( P_{2} \) in \( G \) by \( u, v \). Then \( V(G) = V(P_{n}) \cup V(P_{2}) \) and \( E(G) = \{uv, u_{i}u_{i+1}| 1 \leq i \leq n-1\} \cup \{uu_{i}, vu_{i}| 1 \leq i \leq n\} \). Clearly \( |V(G)| = n + 2 \) and \( |E(G)| = 3n \). Define \( f : V(G) \cup E(G) \rightarrow \{0, 1\} \) as follows:

**Case i.** \( n \) is odd.

Let \( f(u) = f(v) = 0 \), \( f(u_{i}) = 0 \), \( f(uv) = 1 \), \( f(uu_{i}) = f(vu_{i}) = 1 \) for \( 1 \leq i \leq n \) and \( f(u_{i}u_{i+1}) = 0 \) for \( 1 \leq i \leq n-1 \). Then

\[
\text{wt}(u) = f(u) + f(uv) + \sum_{i=1}^{n} f(uu_{i}) = 1 + n \equiv 0 \pmod{2},
\]

\[
\text{wt}(v) = f(v) + f(uv) + \sum_{i=1}^{n} f(vu_{i}) = 1 + n \equiv 0 \pmod{2}
\]

and for \( 1 \leq i \leq n \), \( \text{wt}(u_{i}) = 2 \equiv 0 \pmod{2} \). Also \( n_{f}(0) = n_{f}(1) = 2n + 1 \). Thus, \( |n_{f}(0) - n_{f}(1)| = 0 \).

**Case ii.** \( n = 2k \) and \( k \) is odd.

Let \( f(u) = f(v) = 0 \), \( f(u_{i}) = 1 \), \( f(uv) = 1 \) for \( 1 \leq i \leq n \); \( f(uu_{i}) = f(vu_{i}) = f(u_{i}u_{i+1}) = 1 \) for \( 1 \leq i \leq k \) and \( f(u_{i}u_{i+1}) = 0 \) for \( 1 \leq i < n \). Hence \( \text{wt}(u) = \text{wt}(v) = k + 1 \equiv 0 \pmod{2} \) and \( \text{wt}(u_{i}) = 2 \equiv 0 \pmod{2} \) for \( 1 \leq i \leq n \). Also \( n_{f}(0) = n_{f}(1) = 2n + 1 \). Thus, \( |n_{f}(0) - n_{f}(1)| = 0 \).

**Case iii.** \( n = 2k \) and \( k \) is even.
Let \( f(u) = f(v) = 0, f(u_i) = 1, f(uv) = 1 \), for \( 1 \leq i \leq n \); \( f(uu_i) = f(vu_i) = 1, f(uu_{k+i}) = f(vu_{k+i}) = 0 \) for \( 1 \leq i \leq k \) and \( f(u_iu_{i+1}) = 0 \) for \( 1 \leq i < n \). Hence, \( \text{wt}(u) = \text{wt}(v) = k + 1 \equiv 1 \pmod{2} \), \( \text{wt}(u_i) = 3 \equiv 1 \pmod{2} \) for \( 1 \leq i \leq k \) and \( \text{wt}(u_i) = 1 \) for \( k + 1 \leq i \leq n \). Also \( n_f(0) = n_f(1) = 2n + 1 \). Thus, \( |n_f(0) - n_f(1)| = 0 \).

**Theorem 5.7.** Let \( G(p, q) \) be a TVMC graph with constant \( C = 0 \) where \( p \) is odd. Then \( G + K_{2m} \) is TVMC with \( C = 1 \) if \( m \) is odd and with \( C = 0 \) if \( m \) is even.

**Proof.** Let \( V(G) = \{u_1, u_2, \ldots, u_p\} \), \( V(K_{2m}) = \{v_1, v_2, \ldots, v_m, \ldots, v_{2m}\} \) and \( E(G + K_{2m}) = E(G) \cup \{u_iv_j|1 \leq i \leq p, 1 \leq j \leq 2m\} \). Let \( f \) be the TVMC labeling of \( G \) with \( C = 0 \). Define TVMC labeling \( g \) of \( G + K_{2m} \) as follows:

\[
g(u_jv_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq m, \\
1 & \text{if } m < i \leq 2m.
\end{cases}
\]

When \( m \) is odd,

\[
g(v_i) = \begin{cases} 
1 & \text{if } 1 \leq i \leq m, \\
0 & \text{if } m < i \leq 2m.
\end{cases}
\]

and when \( m \) is even,

\[
g(v_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq m, \\
1 & \text{if } m < i \leq 2m.
\end{cases}
\]

Now we find the weight of the vertices by considering the following two cases:

**Case i.** \( m \) is odd.

For \( v_i \in V(K_{2m}) \),

\[
\text{wt}_{G + K_{2m}}(v_i) = g(v_i) + \sum_{j=1}^{p} g(u_jv_i) = 1 \text{ if } 1 \leq i \leq m,
\]

\[
\text{wt}_{G + K_{2m}}(v_i) = p \equiv 1 \pmod{2} \text{ if } m < i \leq 2m
\]

and for \( u_j \in V(G) \),

\[
\text{wt}_{G + K_{2m}}(u_j) = \text{wt}_G(u_j) + \sum_{i=1}^{m} g(u_jv_i) + \sum_{i=m+1}^{2m} g(u_jv_i)
\]

\[
= \text{wt}_G(u_j) + m \equiv 1 \pmod{2}.
\]

**Case ii.** \( m \) is even.
For $v_i \in V(K_{2m})$,
\[
\begin{align*}
\text{wt}_{G+K_{2m}}(v_i) &= 0 \text{ if } 1 \leq i \leq m, \\
\text{wt}_{G+K_{2m}}(v_i) &= 1 + p \equiv 0 \pmod{2} \text{ if } m < i \leq 2m
\end{align*}
\]
and for $u_j \in V(G)$,
\[
\begin{align*}
\text{wt}_{G+K_{2m}}(u_j) &= \text{wt}_G(u_j) + m \sum_{i=1}^{m} g(u_jv_i) + \sum_{i=m+1}^{2m} g(u_jv_i) \\
&= \text{wt}_G(u_j) + m \equiv 0 \pmod{2}.
\end{align*}
\]
Also $n_g(0) = n_f(0) + m(p + 1)$, $n_g(1) = n_f(1) + m(p + 1)$ and hence $|n_g(0) - n_g(1)| = |n_f(0) - n_f(1)| \leq 1$. Therefore, $G + K_{2m}$ is TVMC.

**Acknowledgement:** The authors sincerely thank the referee for the valuable comments and suggestions for a better presentation of the paper.

**References**


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