Some constructions of supermagic graphs using antimagic graphs

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Abstract. A graph \( G \) is called supermagic if it admits a labelling of the edges by pairwise different consecutive integers such that the sum of the labels of the edges incident with a vertex, the weight of vertex, is independent of the particular vertex. A graph \( G \) is called \((a,1)\)-antimagic if it admits a labelling of the edges by the integers \( \{1, \ldots, |E(G)|\} \) such that the set of weights of the vertices consists of different consecutive integers. In this paper we will deal with the \((a,1)\)-antimagic graphs and their connection to the supermagic graphs. We will introduce three constructions of supermagic graphs using some \((a,1)\)-antimagic graphs.

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§1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If \( G \) is a graph, then \( V(G) \) and \( E(G) \) stand for the vertex set and edge set of \( G \), respectively.

Let a graph \( G \) and a mapping \( f \) from \( E(G) \) into positive integers be given. The index-mapping of \( f \) is the mapping \( f^* \) from \( V(G) \) into positive integers defined by

\[
f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every } v \in V(G),
\]

where \( \eta(v, e) \) is equal to 1 when \( e \) is an edge incident with a vertex \( v \), and 0 otherwise. An injective mapping \( f \) from \( E(G) \) into positive integers is called a magic labelling of \( G \) for an index \( \lambda \) if its index-mapping \( f^* \) satisfies

\[
f^*(v) = \lambda \quad \text{for all } v \in V(G).
\]
A magic labelling $f$ of $G$ is called a *supermagic labelling of $G$* if the set \( \{ f(e) : e \in E(G) \} \) consists of consecutive positive integers. We say that a graph $G$ is *supermagic (magic)* if and only if there exists a supermagic (magic) labelling of $G$.

The concept of magic graphs was introduced by Sedláček [17]. The regular magic graphs are characterized in [4]. Two different characterizations of all magic graphs are given in [14] and [13]. Supermagic graphs were introduced by Stewart [19]. It is easy to see that the classical concept of a magic square of $n^2$ boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic for every positive integer $n \neq 2$ (see also [19]). Stewart [20] characterized supermagic complete graphs. In [10] supermagic regular complete multipartite graphs and supermagic cubes are characterized. In [11] there are given characterizations of magic line graphs of general graphs and supermagic line graphs of regular bipartite graphs. In [16] and [1] supermagic labellings of the Möbius ladders and two special classes of 4-regular graphs are constructed. Some constructions of supermagic labellings of various classes of regular graphs are described in [9] and [10]. In [5] there are established some bounds for number of edges in supermagic graph. More comprehensive information on magic and supermagic graphs can be found in [8].

Let $G$ be a graph. A bijective mapping $f$ from $E(G)$ into the set of integers \( \{ 1, 2, \ldots, |E(G)| \} \) is called an *antimagic labelling of $G$* if the index-mapping $f^*$ is injective, i.e., it satisfies
\[
f^*(v) \neq f^*(u) \quad \text{for every} \quad u, v \in V(G), u \neq v.
\]

The concept of an antimagic labelling was introduced by Hartsfield and Ringel [9]. Bodendiek and Walther [2] introduced the special case of antimagic graphs. For positive integers $a$, $d$, a graph $G$ is said to be \((a,d)\)-antimagic, if it admits an antimagic labelling $f$ such that
\[
\{ f^*(v) : v \in V(G) \} = \{ a, a + d, \ldots, a + (|V(G)| - 1)d \}.
\]

Obviously, $a = \frac{|E(G)|(|E(G)| + 1)}{|V(G)|} - \frac{(|V(G)| - 1)d}{2}$ in this case.

In this paper we will deal with the \((a,1)\)-antimagic graphs and their connection to the supermagic graphs. We will introduce three constructions of supermagic graphs using some \((a,1)\)-antimagic graphs.

## §2. \((a,1)\)-antimagic graphs

It is known that the cycle $C_n$ and the path $P_n$ on $n$ vertices are \((a,1)\)-antimagic if and only if $n$ is odd, see [3]. To find other \((a,1)\)-antimagic graphs we use the edge-magic graphs which were introduced by Kotzig and Rosa [15].
Let $G$ be a graph. A bijection $f : E(G) ∪ V(G) → \{1, 2, \ldots, |E(G)| + |V(G)|\}$ is called an edge-magic total labelling of $G$ if there is a constant $\sigma$ such that

$$f(u) + f(uv) + f(v) = \sigma,$$

for every edge $uv \in E(G)$. Moreover, if the vertices are labelled with the values from the set $\{1, 2, \ldots, |V(G)|\}$ we say that $G$ is a super edge-magic graph.

**Theorem 2.1.** Let $G$ be a 2-regular graph. Then $G$ is super edge-magic if and only if it is $(a, 1)$-antimagic.

**Proof.** Evidently, there is a digraph $\tilde{G}$ which we get from $G$ by an orientation of its edges such that the outdegree of every vertex of $\tilde{G}$ is equal to 1. Let $[u, v]$ denote an arc of $\tilde{G}$.

Suppose that $f$ is a super edge-magic labelling of $G$. Then the labelling $g$, defined by $g(uv) = f(u)$ for every arc $[u, v]$ of $\tilde{G}$, is $(a, 1)$-antimagic.

Assume that $g$ is an $(a, 1)$-antimagic labelling of $G$. Then the labelling $f$, defined by $f(u) = g(uv)$ for every arc $[u, v]$ of $\tilde{G}$ and $f(uv) = (5|V(G)| + 3)/2 - f(u) - f(v)$, is super edge-magic. \qed

According to the previous theorem and a corresponding result for super edge-magic graphs proved in [12] we have the following statement.

**Corollary 2.2.** Let $kG$ be a disjoint union of $k$ copies of a graph $G$. If $G$ is a 2-regular $(a_1, 1)$-antimagic graph, then $kG$ is $(a_2, 1)$-antimagic for every odd positive integer $k$.

Using the previous assertions and results on super edge-magic unions of two cycles (see [6]) we have

**Corollary 2.3.** Let $k, n$ and $m$ be positive integers. For $k$ odd each of the following graphs is $(a, 1)$-antimagic

(i) $kC_n$ if $3 \leq n \equiv 1 \pmod{2}$,

(ii) $k(C_3 \cup C_n)$ if $6 \leq n \equiv 0 \pmod{2}$,

(iii) $k(C_4 \cup C_n)$ if $5 \leq n \equiv 1 \pmod{2}$,

(iv) $k(C_5 \cup C_n)$ if $4 \leq n \equiv 0 \pmod{2}$,

(v) $k(C_m \cup C_n)$ if $6 \leq m \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{2}$, $n \geq m/2 + 2$.

Graphs $G_1, G_2$ form a decomposition of a graph $G$ if $V(G_1) = V(G_2) = V(G)$, $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(G)$. If $G_2$ is an $r$-regular graph then we say that the graph $G$ arose from $G_1$ by adding the $r$-factor $G_2$. At IWOGL held in Herľany 2005 Petr Kovář presented an interesting method
of construction of vertex-magic and antimagic total labellings of graphs (for
definitions see [7]). However, this idea can be also used for \((a, d)\)-antimagic
graphs.

**Theorem 2.4.** Let \(k\) be a positive integer and let \(H\) be a graph which arose
from a graph \(G\) by adding an arbitrary \(2k\)-factor. If \(G\) is an \((a_1, 1)\)-antimagic
graph, then \(H\) is also \((a_2, 1)\)-antimagic.

**Proof.** As every \(2k\)-regular graph is decomposable into \(k\) edge-disjoint
\(2\)-factors, it is sufficient to consider that \(H\) arose from \(G\) by adding a
\(2\)-factor \(F\). Let \(\tilde{F}\) be a digraph which we get from \(F\) by an orientation
of its edges such that the outdegree of every vertex of \(\tilde{F}\) is equal to 1. Let \([u, v]\) denote
an arc of \(\tilde{F}\).

The graph \(G\) is \((a_1, 1)\)-antimagic and so there is its \((a_1, 1)\)-antimagic
labelling \(f\), where \(a_1 = \min\{f^*(v) : v \in V(G)\}\). Consider a mapping \(h : E(H) \rightarrow \{1, 2, \ldots, |E(H)|\}\) defined by

\[
h(e) = \begin{cases} f(e) & \text{if } e \in E(G), \\ a_1 + |E(H)| - f^*(u) & \text{if } e = uv \in E(F) \text{ and } [u, v] \text{ is an arc of } \tilde{F}.
\end{cases}
\]

It is easy to see that \(h\) is a bijection and \(h^*(v) = a_1 + |E(H)| + h(uv)\),
where \([u, v]\) is an arc of \(\tilde{F}\). As \(\{h(e) : e \in E(F)\} = \{|E(G)| + 1, |E(G)| + 2, \ldots, |E(H)|\}\), the labelling \(h\) is \((a_2, 1)\)-antimagic, where
\(a_2 = a_1 + |E(H)| + |E(G)| + 1\).

Let \(n, m\) and \(1 \leq a_1 < \cdots < a_m \leq \left\lfloor \frac{n}{2} \right\rfloor\) be positive integers. A graph
\(C_n(a_1, \ldots, a_m)\) with the vertex set \(\{v_1, \ldots, v_n\}\) and the edge set \(\{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}\), the indices are being taken modulo \(n\), is called a *circulant
graph*. Clearly, \(C_n(a_1, \ldots, a_m)\) arose from \(C_n(a_m)\) by adding a \(2(m - 1)\)-factor.
Moreover, if \(n\) is odd, then \(C_n(a_m)\) is an \((a, 1)\)-antimagic graph because it is
isomorphic to \(kC_r\), where \(k\) and \(r\) are odd. Therefore, we have immediately

**Corollary 2.5.** Every circulant graph of odd order is \((a, 1)\)-antimagic.

The cycle of odd order is \((a, 1)\)-antimagic and every regular Hamiltonian
graph arose from its Hamilton cycle by adding a factor, so

**Corollary 2.6.** Every \(2r\)-regular Hamiltonian graph of odd order is \((a, 1)\)-
antimagic.

Any graph of order \(n\) with minimum degree at least \(n/2\) is Hamiltonian,
thus we get

**Corollary 2.7.** Let \(G\) be a \(2r\)-regular graph of odd order \(n\). If \(n < 4r\), then
\(G\) is \((a, 1)\)-antimagic.
§3. Supermagic graphs

For any graph $G$ we define a graph $G^{\circ}$ by $V(G^{\circ}) = \bigcup_{v \in V(G)} \{v^0, v^1\}$ and $E(G^{\circ}) = E_1(G^{\circ}) \cup E_2(G^{\circ})$, where $E_1(G^{\circ}) = \bigcup_{u,v \in E(G)} \{v^0u^1, v^1u^0\}$ and $E_2(G^{\circ}) = \bigcup_{v \in V(G)} \{v^0v^1\}$.

**Theorem 3.1.** Let $G$ be an $(a,1)$-antimagic $2r$-regular graph. Then $G^{\circ}$ is a supermagic graph.

*Proof.* Put $n := |V(G)|$. As $G$ is a $2r$-regular graph, every its component is Eulerian. Therefore, there is a digraph $	ilde{G}$ which we get from $G$ by an orientation of its edges such that the outdegree (and also the indegree) of every vertex of $	ilde{G}$ is equal to $r$. By $[u,v]$ we denote an arc of $	ilde{G}$ and by $N^+(v)$, $N^-(v)$ the outneighbourhood, inneighbourhood of a vertex $v$ in $	ilde{G}$, respectively.

Let $f : E(G) \rightarrow \{1, 2, \ldots, rn\}$ be an $(a,1)$-antimagic labelling of $G$. Consider the bijection $g : E_1(G^{\circ}) \rightarrow \{1, 2, \ldots, 2rn\}$ given by

$$g(u^iv^j) = \begin{cases} f(uv) & \text{if } i = 0, j = 1, \\ f(uv) + rn & \text{if } i = 1, j = 0, \end{cases}$$

for every arc $[u,v]$ of $	ilde{G}$.

For its index-mapping we have

$$g^*(v^0) = \sum_{w \in N^+(v)} g(v^0w^1) + \sum_{u \in N^-(v)} g(u^1v^0)$$

$$= \sum_{w \in N^+(v)} f(uw) + \sum_{u \in N^-(v)} (f(uv) + rn) = f^*(v) + r^2 n$$

for every vertex $v^0 \in V(G^{\circ})$. Similarly, we have $g^*(v^1) = f^*(v) + r^2 n$ for every vertex $v^1 \in V(G^{\circ})$. Thus $g^*(v^0) = g^*(v^1) = f^*(v) + r^2 n$ for every vertex $v \in V(G)$. As $f$ is an $(a,1)$-antimagic labelling, the set $\{f^*(v) : v \in V(G)\}$ consists of consecutive integers. It means that the bijection $h : E(G^{\circ}) \rightarrow \{1, 2, \ldots, (2r + 1)n\}$, given by

$$h(u^iv^j) = g(u^iv^j) \quad \text{for } u^iv^j \in E_1(G^{\circ}),$$

$$h(v^0v^1) = \frac{2rn(r + 1) + (2r + 1)(n + 1)}{2} - f^*(v) \quad \text{for } v \in V(G),$$

is a supermagic labelling of $G^{\circ}$. \hfill \Box
Note, that $C_n^m$ is a graph isomorphic to either the Möbius ladder $M_{2n}$, for $n$ odd, or the graph of $n$-side prism $S_n$, for $n$ even. Moreover, for the disjoint union of graphs $G_1$ and $G_2$ it holds $(G_1 \cup G_2)^{\infty} = G_1^{\infty} \cup G_2^{\infty}$. According to Theorem 3.1 and Corollary 2.3 we have

**Corollary 3.2.** Let $k$, $n$ and $m$ be positive integers. For $k$ odd the following graphs are supermagic

(i) $kM_{2n}$ when $3 \leq n \equiv 1 \pmod{2}$,

(ii) $k(M_6 \cup S_n)$ when $6 \leq n \equiv 0 \pmod{2}$,

(iii) $k(S_4 \cup M_{2n})$ when $5 \leq n \equiv 1 \pmod{2}$,

(iv) $k(M_{10} \cup S_n)$ when $4 \leq n \equiv 0 \pmod{2}$,

(v) $k(S_m \cup M_{2n})$ when $6 \leq m \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{2}$, $n \geq m/2 + 2$.

Similarly, using Theorem 3.1 and Corollaries 2.5, 2.6 and 2.7 we get

**Corollary 3.3.** Let $G$ be a $2r$-regular graph of odd order $n$. If $G$ is circulant, Hamiltonian or $n < 4r$, then $G^{\infty}$ is a supermagic graph.

One can see that $G^{\infty}$ is isomorphic to the Cartesian product $G \times K_2$ whenever $G$ is a bipartite graph. However, a regular bipartite graph of even degree is never $(a,1)$-antimagic. So, in the next theorem we describe another construction of supermagic Cartesian products.

**Theorem 3.4.** Let $G$ be an $(a,1)$-antimagic graph decomposable into two edge-disjoint $r$-factors. Then $G \times K_2$ is a supermagic graph.

**Proof.** Suppose that $F^1_1$, $F^2_1$ are edge-disjoint $r$-factors which form a decomposition of $G$ and $f : E(G) \rightarrow \{1,2,\ldots, rn\}$, where $n = |V(G)|$, is an $(a,1)$-antimagic labelling of $G$.

We can denote the vertices of $G \times K_2$ by $v_i, i \in \{1,2\}, v \in V(G)$, in such a way that the vertices $\{v_i : v \in V(G)\}$ induce a subgraph $G_i$ isomorphic to $G$. So, $G \times K_2$ consists of subgraphs $G_1$, $G_2$ and $n$ edges $v_1v_2$ for all $v \in V(G)$. By $F^j_i, i \in \{1,2\}, j \in \{1,2\}$, we denote the factor of $G_i$ corresponding to $F^j_i$.

Consider the bijection $g : E(G_1 \cup G_2) \rightarrow \{1,2,\ldots, 2rn\}$ given by

$$g(e) = \begin{cases} f(e) & \text{if } e \in F^1_1 \text{ or } e \in F^2_1, \\ f(e) + rn & \text{if } e \in F^1_2 \text{ or } e \in F^2_2. \end{cases}$$
Theorem 3.6. Let $G$ be a $r$-regular graph of odd order $n$. If $G$ is circulant, Hamiltonian or $n < 8r$, then $G \oplus K_2$ is a supermagic graph.

As every 4r-regular graph is decomposable into two edge-disjoint 2r-factors, immediately from Theorem 3.4 and Corollaries 2.5, 2.6 and 2.7 we get

Corollary 3.5. Let $G$ be a 4r-regular graph of odd order $n$. If $G$ is circulant, Hamiltonian or $n < 8r$, then $G \oplus K_2$ is a supermagic graph.

Finally we describe a construction of supermagic joins $G \oplus K_1$. In [18] there are given some conditions for the existence of such graphs.

Theorem 3.6. Let $G$ be an $(a, 1)$-antimagic r-regular graph of order $n$. If $(n - r - 1)$ is a divisor of the non-negative integer $a + n(1 + r - \frac{n+1}{2})$, then the join $G \oplus K_1$ is a supermagic graph.

Proof. Put $\lambda_1 := a + n(1 + r)$ and $\lambda_2 := \frac{a(n+1)}{2}$. According to the assumption there is a non-negative integer $p$ such that $\lambda_1 - \lambda_2 = p(n - r - 1)$ (thus $(r + 1)p + \lambda_1 = np + \lambda_2$). Let $f$ be an $(a, 1)$-antimagic labelling of $G$. The join $G \oplus K_1$ is obtained from $G$ by adding the vertex $w$ and the edges $vw$ for all $v \in V(G)$.

Consider the mapping $h$ from $E(G \oplus K_1)$ into positive integers given by

$$h(e) = \begin{cases} p + n + f(e) & \text{if } e \in E(G), \\ p + n + a - f^*(v) & \text{if } e = vw \text{ for } v \in V(G). \end{cases}$$
Evidently, \( \{h(wv) : v \in V(G)\} = \{p + 1, p + 2, \ldots, p + n\} \) and \( \{h(e) : e \in E(G)\} = \{p + n + 1, p + n + 2, \ldots, p + n + |E(G)|\} \). Thus, the set \( \{h(e) : e \in E(G \oplus K_1)\} \) consists of consecutive positive integers. Moreover, \( h^*(w) = np + \lambda_2 \) and \( h^*(v) = (r+1)p + \lambda_1 \) for all \( v \in V(G) \). Therefore, \( h \) is a supermagic labelling of \( G \oplus K_1 \).

Using the divisibility it is not difficult to check the assumptions of Theorem 3.6 for given values \( n \) and \( r \). Thus we have

**Corollary 3.7.** Let \( n \) and \( r \) be positive integers such that one of the following conditions is satisfied:

(i) \( 5 \leq n \equiv 1 \pmod{2} \) and \( r = n - 3 \),

(ii) \( 11 \leq n \equiv 1 \pmod{2} \) and \( r = n - 7 \),

(iii) \( 8 \leq n \equiv 0 \pmod{4} \) and \( r = \frac{n}{2} - 1 \),

(iv) \( 11 \leq n \equiv 3 \pmod{8} \) and \( r = n - 5 \),

(v) \( 12 \leq n \equiv 4 \pmod{8} \) and \( r = n - 3 \),

(vi) \( 12 \leq n \equiv 4 \pmod{8} \) and \( r = n - 7 \),

(vii) \( 13 \leq n \equiv 5 \pmod{8} \) and \( r = n - 5 \).

If \( G \) is an \((a,1)\)-antimagic \( r \)-regular graph of order \( n \), then the join \( G \oplus K_1 \) is supermagic.

Immediately from Corollaries 2.7 and 3.7 we get

**Corollary 3.8.** Let \( G \) be any \((n-3)\)-regular \((n-7)\)-regular) graph of odd order \( n \geq 7 \) \((n \geq 15)\). Then \( G \oplus K_1 \) is a supermagic graph.

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**References**


CONSTRUCTIONS OF SUPERMAGIC GRAPHS


