Degree-Sum Conditions for Graphs to Have 2-Factors with Cycles Through Specified Vertices

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Abstract. Let \( k \geq 2 \) and \( n \geq 1 \) be integers, let \( G \) be a graph of order \( n \) with minimum degree at least \( k + 1 \). Let \( v_1, v_2, \ldots, v_k \) be \( k \) distinct vertices of \( G \), and suppose that there exist \( k \) vertex disjoint cycles \( C_1, \ldots, C_k \) in \( G \) such that \( v_i \in V(C_i) \) for each \( 1 \leq i \leq k \). Suppose further that the minimum value of the sum of the degrees of two nonadjacent distinct vertices is greater than or equal to \( n + \frac{k-1}{3} \). Under these assumptions, we show that there is a 2-factor of \( G \) with \( k \) cycles \( D_1, D_2, \ldots, D_k \) such that \( v_i \in V(D_i) \) for each \( 1 \leq i \leq k \).

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§1. Introduction

All graphs considered in this paper are finite simple undirected graphs with no loops and no multiple edges. For a graph \( G \), we let \( V(G) \) and \( E(G) \) denote the set of vertices and edges of \( G \), respectively. For a vertex \( v \) of \( G \), we let \( d_G(v) \) denote the degree of \( v \) in \( G \). We define \( \delta(G) \) to be the minimum degree of \( G \), and \( \sigma_2(G) \) to be the minimum value of the sum of the degrees of two nonadjacent distinct vertices.

The following theorem appears in [1]:

**Theorem A.** Let \( k \) and \( n \) be positive integers with \( n \geq 3k \), and let \( G \) be a graph of order \( n \). Suppose that one of the following four conditions is satisfied:

a) \( n = 3k \) and \( \delta(G) \geq \frac{7k-2}{3} \);

b) \( 3k + 1 \leq n \leq 4k \) and \( \delta(G) \geq \frac{2n+k-3}{3} \);
c) \( 4k \leq n \leq 6k - 3 \) and \( \delta(G) \geq 3k - 1 \); or

d) \( n \geq 6k - 3 \) and \( \delta(G) \geq \frac{n}{2} \).

Then for any \( k \) distinct vertices \( v_1, \cdots, v_k \), there is a 2-factor of \( G \) with \( k \) cycles \( C_i, 1 \leq i \leq k \) such that \( v_i \in V(C_i) \) for each \( 1 \leq i \leq k \).

In [1], Theorem A is derived as an immediate corollary of the following two theorems:

**Theorem B.** Let \( k \) and \( n \) be positive integers with \( n \geq 3k \), and let \( G \) be a graph of order \( n \). Suppose that one of the conditions a) through d) in Theorem A is satisfied. Then for any \( k \) distinct vertices \( v_1, \cdots, v_k \), there exist \( k \) vertex disjoint cycles \( C_i, 1 \leq i \leq k \), such that \( |V(C_i)| \leq 5 \) and \( v_i \in V(C_i) \) for each \( 1 \leq i \leq k \).

**Theorem C.** Let \( k \) and \( n \) be positive integers, and let \( G \) be a graph of order \( n \) such that \( \sigma_2(G) \geq n, \delta(G) \geq k + 1 \) and \( \sigma_2(G) + \delta(G) \geq n + 3k - 2 \). Let \( v_1, \cdots, v_k \) be distinct vertices of \( G \), and suppose that there exist \( k \) vertex disjoint cycles \( C_i, 1 \leq i \leq k \), such that \( v_i \in V(C_i) \) for each \( 1 \leq i \leq k \). Then there is a 2-factor of \( G \) with \( k \) cycles \( D_i, 1 \leq i \leq k \), such that \( v_i \in V(D_i) \) for each \( 1 \leq i \leq k \).

This paper is concerned with Theorem C. In Theorem C, the condition \( \sigma_2(G) + \delta(G) \geq n + 3k - 2 \) is of technical nature, and it has been conjectured that Theorem C holds even if we drop the condition \( \sigma_2(G) + \delta(G) \geq n + 3k - 2 \). This conjecture has partially been settled affirmatively by the following two theorems proved in [2] (Theorem D says that if we regard a path of length 0 or 1 as a cycle of length 1 or 2, respectively, then the conjecture is true; Theorem E says that if \( n \geq 6k - 4 \), then the conjecture is true):

**Theorem D.** Let \( k \) and \( n \) be positive integers, and let \( G \) be a graph of order \( n \). Suppose that \( \sigma_2(G) \geq n \) and \( \delta(G) \geq k + 1 \). Then for any \( k \) distinct vertices \( v_1, \cdots, v_k \), there is a spanning subgraph of \( G \) with \( k \) components \( C_i, 1 \leq i \leq k \), such that for each \( i \), \( v_i \in V(C_i) \) and \( C_i \) is either a cycle or a path of length 0 or 1.

**Theorem E.** Let \( k \) and \( n \) be positive integers with \( n \geq 6k - 4 \), and let \( G \) be a graph of order \( n \) such that \( \sigma_2(G) \geq n \) and \( \delta(G) \geq k + 1 \). Let \( v_1, \cdots, v_k \) be distinct vertices of \( G \), and suppose that there exist \( k \) vertex disjoint cycles \( C_i, 1 \leq i \leq k \), such that \( v_i \in V(C_i) \) for each \( 1 \leq i \leq k \). Then there is a 2-factor of \( G \) with \( k \) cycles \( D_i, 1 \leq i \leq k \), such that \( v_i \in V(D_i) \) for each \( 1 \leq i \leq k \).
In this paper, we give the following (somewhat negative) solution to the conjecture (as we shall see in the second paragraph following the statement of Theorem 1, the condition $\sigma_2(G) \geq n + \frac{k-4}{3}$ in Theorem 1 is best possible, which means that if we drop the condition $\sigma_2(G) + \delta(G) \geq n + 3k - 2$ in Theorem C, then we need to replace the condition $\sigma_2(G) \geq n$ by the stronger condition $\sigma_2(G) \geq n + \frac{k-4}{3}$):

**Theorem 1.** Let $k \geq 2$ and $n \geq 1$ be integers, and let $G$ be a graph of order $n$ such that $\sigma_2(G) \geq n + \frac{k-4}{3}$ and $\delta(G) \geq k+1$. Let $v_1, \ldots, v_k$ be distinct vertices of $G$, and suppose that there exist $k$ vertex disjoint cycles $C_i, 1 \leq i \leq k$, such that $v_i \in V(C_i)$ for each $1 \leq i \leq k$. Then there is a 2-factor of $G$ with $k$ cycles $D_i, 1 \leq i \leq k$, such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$.

Theorem 1 does not hold for $k = 1$. To see this, let $l \geq 2$ be an integer, and consider a complete bipartite graph $G = K(l, l+1)$. Then $\sigma_2(G) = 2l = |V(G)| - 1 = |V(G)| + \frac{k-4}{3}$ and $\delta(G) = l \geq 2 = k+1$, but $G$ does not have a 2-factor. Also the condition $\delta(G) \geq k+1$ is best. To verify this, let $k, n$ be integers with $k \geq 2$ and $n \geq 3k + 1$, and define a graph $G$ of order $n$ as follows: let $L$ be a complete graph of order $n - 1$ containing specified distinct vertices $v_1, v_2, \ldots, v_k$, let $u$ be a vertex with $u \notin V(L)$, and let $V(G) = V(L) \cup \{u\}$ and $E(G) = E(L) \cup \{uv_i | 1 \leq i \leq k\}$. Then $\delta(G) = d_G(u) = k$, $\sigma_2(G) = (n-2) + k = n + k - 2 \geq n + \frac{k-4}{3}$, and there exist $k$ vertex disjoint cycles $C_1, \ldots, C_k$ in $L$ (so in $G$) such that $v_i \in V(C_i)$ for each $1 \leq i \leq k$. But $G$ does not contain a 2-factor with $k$ cycles $D_1, D_2, \ldots, D_k$ such that $v_i \in V(D_i)$ for each $1 \leq i \leq k$ because any cycle containing $u$ must passes through at least two vertices in $\{v_1, v_2, \ldots, v_k\}$.

Finally we verify that the condition $\sigma_2(G) \geq n + \frac{k-4}{3}$ is best for $3k + 2 \leq n \leq \lceil \frac{14}{3}k \rceil$. Let $k, n$ be integers with $k \geq 2$ and $3k + 2 \leq n \leq \lceil \frac{14}{3}k \rceil$, and write $n = 3k + h$. We define a graph $G$ of order $n$ as follows. Let $H$ be a complete graph of order $h$. Let $m = \lceil \frac{14}{3} \rceil$ and let $L$ be the graph of order $3k$ with vertex set

$$V(L) = \{v_i | 1 \leq i \leq k\} \cup \{u_i | 1 \leq i \leq k\} \cup \{w_i | 1 \leq i \leq k\}$$

such that the complement graph $\overline{L}$ of $L$ satisfies

$$E(\overline{L}) = \{w_iw_j | 1 \leq i < j \leq m\} \cup \{w_iw_k | 1 \leq i \leq m\} \cup \{v_iw_j | m + 1 \leq i \leq k, j \neq i\} \cup \{v_iw_j | 1 \leq i \leq m, m + 1 \leq j \leq k - 1\}$$

(Figure 1). Define $G$ by $V(G) = V(H) \cup V(L)$ and $E(G) = E(H) \cup E(L) \cup \{hu_i | h \in V(H), 1 \leq i \leq k\}$. 


Figure 1: Dotted lines stand for the edges of $\mathcal{L}$ while solid lines stand for some of edges of $L$ for the case where $k = 5$.

Then

$$d_G(h) = h + k - 1 \quad (\text{for } h \in V(H)),$$

$$d_G(u_i) = d_L(u_i) + |V(H)| = \begin{cases} 
  n - 1 & (\text{for } 1 \leq i \leq m), \\
  n - k & (\text{for } m + 1 \leq i \leq k), 
\end{cases}$$

$$d_G(v_i) = d_L(v_i) = \begin{cases} 
  k + 2m & (\text{for } 1 \leq i \leq m), \\
  2k + m & (\text{for } m + 1 \leq i \leq k), 
\end{cases}$$

$$d_G(w_i) = d_L(w_i) = 3k - m - 1 \quad (\text{for } 1 \leq i \leq k),$$

and hence $\delta(G) \geq k + 1$. Next we verify that $\sigma_2(G) \geq n + \frac{k-4}{3} - 1$. Let $\varepsilon_k = 0, 2, 1$ and $c_k = 3, 4, 2$ according as $k \equiv 0, 1, 2 \pmod{3}$. Then $m = \frac{2k-\varepsilon_k}{3}$, $\frac{k-\varepsilon_k}{3}$ is an integer, the condition $\sigma_2(G) \geq n + \frac{k-4}{3} - 1$ is equivalent to $\sigma_2(G) \geq \left\lceil \frac{10k-\varepsilon_k}{3} \right\rceil - n \geq 0$ and

(i) $d_G(h) + d_G(v_i) - \left( n + \frac{k-\varepsilon_k}{3} - 1 \right) \geq \frac{\varepsilon_k-2\varepsilon_k}{3} \geq 0$ and $d_G(h) + d_G(w_i) - \left( n + \frac{k-\varepsilon_k}{3} - 1 \right) \geq 0$ for any $h \in V(H)$ and any $i$ with $1 \leq i \leq k$,

(ii) $d_G(w_i) + d_G(w_j) - \left( n + \frac{k-\varepsilon_k}{3} - 1 \right) = \frac{13k+2\varepsilon_k+c_k-3}{3} - n \geq \left\lceil \frac{13k}{3} \right\rceil - n \geq 0$ for any $i$ and $j$ with $1 \leq i < j \leq m$, or $1 \leq i \leq m$ and $j = k$, for $m = \frac{2k-\varepsilon_k}{3}$.
(iii) \( d_G(u_i) + d_G(v_j) - \left(n + \frac{k-c_k}{3} - 1\right) \geq k + \frac{c_k-2c_k}{3} + 1 \geq k + 1 > 0 \) for any \( i \) with \( m + 1 \leq i \leq k \) and any \( j \) with \( j \neq i \), and

(iv) \( d_G(v_i) + d_G(w_j) - \left(n + \frac{k-c_k}{3} - 1\right) = \frac{13k+c_k-c_k}{3} - n \geq \left\lceil \frac{13k+1}{3} \right\rceil - n > 0 \) for any \( i \) with \( 1 \leq i \leq m \) and any \( j \) with \( m + 1 \leq j \leq k - 1 \).

Consequently, we have \( \sigma_2(G) \geq n + \frac{k-c_k}{3} - 1 \). Also \( G \) contains vertex disjoint cycles \( C_i = u_iv_iw_iu_i, 1 \leq i \leq k \), such that \( v_i \in V(C_i) \) for each \( i \). However, \( G \) does not contain a 2-factor with \( k \) cycles \( D_1, D_2, \ldots, D_k \) such that \( v_i \in V(D_i) \) for each \( i \). First note that each \( D_i \) must contain exactly three vertices in \( V(L) \). Let \( D_{i_1} \) be a cycle containing a vertex in \( V(H) \). Then \( D_{i_1} \) contains (exactly) two vertices in \( \{u_1, \ldots, u_k\} \), and hence there is a cycle \( D_{i_2} \) that consists of \( v_{i_2} \) and two vertices \( w_{j_1} \) and \( w_{j_2} \), where \( j_1 \neq j_2 \). For each \( i \) with \( m + 1 \leq i \leq k \), \( u_i \) is the only vertex in \( \{v_1, \ldots, v_k\} \) that is adjacent to \( u_i \), and hence \( u_i \) and \( v_i \) are in the same cycle. Therefore

\[
1 \leq i_2 \leq m.
\]

On the other hand, \( w_{i_1}w_{j_1} \in E(G) \) only when at least one of \( i \) and \( j \) is in \( \{m + 1, \ldots, k - 1\} \). Thus we may assume \( m + 1 \leq j_1 \leq k - 1 \). But then \( w_{j_1}v_{i_2} \notin E(G) \) by (1.1) and the construction of \( G \), a contradiction.

Our notation is standard with the possible exception of the following:
Let \( G \) be a graph. For a subset \( U \) of \( V(G) \), we let \( \langle U \rangle_G \) denote the subgraph of \( G \) induced by \( U \). For a vertex \( v \) of \( G \), we denote by \( N_G(v) \) the set of neighbours of \( v \). For a vertex \( v \) of \( G \) and for a subgraph \( H \) of \( G \) with \( v \notin V(H) \), we let \( N_H(v) = N_G(v) \cap V(H) \) and \( d_H(v) = |N_H(v)| \). Furthermore, for subgraphs \( H \) and \( K \) of \( G \) with \( V(H) \cap V(K) = \emptyset \), we let \( N_H(K) = \cup_{v \in V(K)} N_H(v) \). Let \( C = u_0u_1 \cdots u_{m-1}u_0 \) be a cycle (indices are to be read modulo \( m \)). For two vertices \( u_i, u_j \) on \( C \) with \( i \leq j \leq i + m - 1 \), we let \( C[u_i, u_j] \) and \( C(u_i, u_j) \) denote the paths \( u_iu_{i+1} \cdots u_j \) and \( u_{i+1}u_{i+2} \cdots u_{j-1} \), respectively (if \( j = i \) or \( j = i + 1 \), \( C(u_i, u_j) \) denotes an empty path).

\section{Proof of Theorem 1}

Throughout this section, let \( k, n, G, v_i \) and \( C_i \) be as in Theorem 1. We may assume we have chosen \( k \) cycles \( C_i \) so that \( \sum_{i=1}^{k} |V(C_i)| \) is maximum. Let \( L = \langle \bigcup_{i=1}^{k} V(C_i) \rangle_G \), \( l = |V(L)| \) and \( H = G - L \). If \( V(H) = \emptyset \), then there is nothing to be proved. Thus suppose that

\[
V(H) \neq \emptyset.
\]
Let $H_0$ be a connected component of $H$. We shall obtain a contradiction in a series of claims. The first five claims, Claims 1 through 5, are essentially proved in [1], but we include their proofs for the convenience of the reader.

**Claim 1.** Let $D_i$, $1 \leq i \leq k$, be vertex disjoint subgraphs of $L$ such that $D_i$ is a path (we allow $D_i$ to consist of a single vertex), and $D_i$ is a cycle for each $2 \leq i \leq k$. Suppose that $D_i$ contains exactly one vertex in $\{v_1, \ldots, v_k\}$ for each $1 \leq i \leq k$, and write $V(D_i) \cap \{v_1, \ldots, v_k\} = \{v_j\}$. Suppose further that $d_L(u) \geq l - \frac{1}{2} |\cup_{i=1}^{k} V(D_i)|$ for all $u \in V(L) - \cup_{i=1}^{k} V(D_i)$. Then there are vertex disjoint subgraphs $D_i^*$, $1 \leq i \leq n$, of $L$ such that $D_i^*$ is a path, $D_i^*$ is a cycle for each $2 \leq i \leq k$, $v_j \in V(D_i^*)$ for each $1 \leq i \leq k$, and $\cup_{i=1}^{k} V(D_i^*) = V(L)$, and such that $D_i^*$ has the same initial vertex and the same terminal vertex as $D_i$ (thus if $D_i$ consists of a single vertex, then $D_i^* = D_i$).

**Proof.** Choose vertex disjoint subgraphs $D_i^*$, $1 \leq i \leq k$, of $L$ satisfying the same conditions as the $D_i$ so that $\sum_{i=1}^{k} |V(D_i^*)|$ is as large as possible, subject to the condition that $V(D_i^*) \supseteq V(D_i)$ for each $1 \leq i \leq k$ and $D_i^*$ has the same initial vertex and the same terminal vertex as $D_i$. Let $A = \cup_{i=1}^{k} V(D_i^*)$. Suppose that $A \neq V(L)$, and take $u \in V(L) - A$. Then $d_L(u) \geq l - \frac{1}{2} |\cup_{i=1}^{k} V(D_i)| \geq l - \frac{1}{2} |A|$ by assumption, and hence $d_{(A)}(u) = d_L(u) - d_{L-A}(u) \geq l - \frac{1}{2} |A| - (l - |A| - 1) = \frac{1}{2} |A| + 1$. Consequently there exist $v, v' \in N_G(u)$ such that $v$ and $v'$ are consecutive on some $D_i^*$, $1 \leq i \leq k$. Inserting $u$ into $D_i^*$, we get a contradiction to the maximality of $|A|$. Thus $A = V(L)$, as desired.

**Claim 2.**

(i) $|N_{C_i}(H_0)| \leq 1$ for each $1 \leq i \leq k$.

(ii) $H = H_0$.

**Proof.** (i) First note that for each $1 \leq i \leq k$,

(2.2) no two vertices in $N_{C_i}(H_0)$ are consecutive on $C_i$

because of the maximality of $\sum_{i=1}^{k} |V(C_i)|$. By way of contradiction, suppose $|N_{C_i}(H_0)| \geq 2$ for some $i$, say $i = 1$. Take $u_1, u_2 \in V(C_1)$ and $h_1, h_2 \in V(H_0)$ (we allow the possibility that $h_1 = h_2$) such that $u_i h_i \in E(G)$ for $i = 1, 2$, $V(C_1(u_1, u_2)) \neq \emptyset$ and such that $N_{H_0}(C_1(u_1, u_2)) = \emptyset$. Let $P = C_1[u_2, u_1]$ and let $Q$ be a path in $H_0$ having $h_1$ and $h_2$ as its initial vertex and its terminal vertex, respectively. Then $C = PQ u_2$ is a cycle containing $v_1$. Let $R = C_1(u_1, u_2)$ and $r = |V(R)|$, and take $w \in V(R)$ (note that $V(R) \neq \emptyset$ by
which implies

\[ N(V) \]

Thus applying Claim 1 with \( D \), we obtain

\[ d_G(w) + d_G(h_1) \geq n + \left\lceil \frac{k-4}{3} \right\rceil \geq n. \]

Consequently

\[
d_L(w) \geq d_G(w) - |V(H) - V(H_0)| \geq l - \frac{1}{2}(l - r) + \frac{1}{2}
\]

and hence it follows from Claim 1 that there exist vertex disjoint subgraphs \( P', C_2', \ldots, C_k' \) of \( L \) such that \( P' \) is a path with \( v_1 \in V(P') \) joining \( u_2 \) and \( u_1, C_i' \) is a cycle with \( v_i \in V(C_i') \) for each \( 2 \leq i \leq k \), and \( V(P') \cup (\cup_{i=2}^k V(C_i')) = V(L) \). Set \( C'_1 = P'Qu_2 \). Then \( C'_1, C'_2, \ldots, C'_k \) are vertex disjoint cycles such that \( v_i \in V(C'_i) \) for each \( 1 \leq i \leq k \) and such that \( \sum_{i=1}^k |V(C'_i)| = \sum_{i=1}^k |V(C_i)| + |V(Q)|. \) This contradicts the maximality of \( \sum_{i=1}^k |V(C_i)| \).

(ii) Suppose \( H \neq H_0 \). Take \( h \in V(H_0) \) and \( h' \in V(H) - V(H_0) \). Then \( n \leq d_G(h) + d_G(h') < (|H_0| - 1 + d_L(h)) + (|H| - |H_0| - 1 + d_L(h')) = d_L(h) + d_L(h') + |H| - 2, \) and hence \( d_L(h) + d_L(h') \geq n - |H| + 2 = l + 2 \geq 3k + 2. \) On the other hand, it follows from (i) of this claim that \( d_L(h) + d_L(h') \leq k + k = 2k. \) Consequently \( 3k + 2 \leq 2k, \) a contradiction. \( \Box \)

Set \( S = \{ u \in V(L) \mid d_H(u) \geq 2 \} \) and let \( s = |S| \). By Claim 2 (i), \( |S \cap V(C_i)| \leq 1 \) for each \( i \in \{ 1, \ldots, k \} \). We may assume \( S \cap V(C_i) = \{ u_i \} \) for each \( i \in \{ 1, \ldots, s \} \) and \( S \cap V(C_i) = \emptyset \) for each \( i \in \{ s+1, \ldots, k \} \).

Claim 3.

(i) \( u_i \neq v_i \) for each \( i \in \{ 1, \ldots, s \} \).

(ii) \( |V(H)| > k - s. \)

Proof. (i) Suppose \( v_i = u_i \) for some \( i \in \{ 1, \ldots, s \} \), say \( i = 1 \). Take \( h_1, h_2 \in N_H(v_1) \) with \( h_1 \neq h_2 \). Since \( H \) is connected by Claim 2 (ii), there is a path \( Q \) in \( H \) connecting \( h_1 \) and \( h_2 \). Take \( u \in V(C_1) - \{ v_1 \} \). By Claim 2 (i), \( u \) is adjacent to no vertex in \( V(H) \). Hence \( n \leq d_G(u) + d_G(h_1) = d_L(u) + d_G(h_1), \) which implies \( d_L(u) \geq n - (|V(H)| - 1 + k) = l - (k - 1) > l - \frac{1}{2} \sum_{i=2}^k |V(C_i)| \). Thus applying Claim 1 with \( D_1 = v_1 \), we obtain \( k - 1 \) vertex disjoint cycles \( C''_i(2 \leq i \leq k) \) in \( L \) such that \( v_i \in V(C''_i) \) for each \( 2 \leq i \leq k \) and \( \{ v_1 \} \cup (\cup_{i=2}^k V(C''_i)) = V(L) \). Set \( C''_1 = v_1Qv_1 \). Then the \( C''_i(1 \leq i \leq k) \) are vertex disjoint cycles such that \( v_i \in V(C''_i) \) for each \( 1 \leq i \leq k \), which contradicts the maximality of \( \sum_{i=1}^k |V(C_i)| \).

(ii) Suppose \( |V(H)| \leq k - s. \) Let \( E(H, L) \) denote the set of edges which have one endvertex in \( V(H) \) and the other in \( V(L). \) Since \( \delta(G) \geq k + 1, \)
|V(H)|(|k+1−(|V(H)|−1)| ≤ |E(H, L)| ≤ s|V(H)|+(k−s) = s(|V(H)|−1)+k, and hence |V(H)|(|k+2−|V(H)|)| ≤ (k−|V(H)|)(|V(H)|−1) + k by the assumption that |V(H)| ≤ k − s. This implies 2|V(H)| ≤ |V(H)|, which contradicts (2.1).

Recall $\sigma_2(G) ≥ n + \frac{k-c_k}{3}$. Let $c_k = 3, 4, 2$ according to whether $k \equiv 0, 1, 2 \pmod{3}$. Then we have

\begin{equation}
\sigma_2(G) \geq n + \frac{k-c_k}{3}
\end{equation}

because $\sigma_2(G)$ is an integer.

**Claim 4.** $d_L(u) ≥ l - s + \frac{k-c_k}{3} + 1$ for all $u ∈ V(L) - N_L(H)$.

**Proof.** Since $\sum_{h \in V(H)} d_G(h) ≤ |V(H)|(|V(H)|−1) + s|V(H)| + k − s$, there exists $h ∈ V(H)$ such that $d_G(h) ≤ |V(H)|−1 + s + \frac{k-s}{|V(H)|}$. In view of Claim 3(ii), we have $d_G(h) ≤ |V(H)|−1 + s$. Thus for every $u ∈ V(L) - N_L(H)$,

$$d_L(u) = d_G(u) ≥ n + \frac{k-c_k}{3} - d_G(h) ≥ l - s + \frac{k-c_k}{3} + 1$$

by (2.3).

For each $u ∈ V(L) - N_L(H)$, $d_L(u) ≥ l - s + 1 = |(V(L) - \{u\}) - S| + 2$ by Claim 4, and hence $|N_G(u) ∩ S| ≥ 2$. In view of Claims 2(i) and 3(i), this in particular implies that $s ≥ 2$, and

\begin{equation}
N_G(v_i) ∩ (S - \{u_i\}) ≠ \emptyset
\end{equation}

for each $i ∈ \{1, \cdots, s\}$.

**Claim 5.** There exist no vertex disjoint subgraphs $P, C_i, 2 ≤ i ≤ k$, in $L$ such that

\begin{equation}
P \text{ is a path joining two distinct vertices in } \{u_1, \cdots, u_s\}, C_i \text{ is a cycle for each } 2 ≤ i ≤ k, \text{ each of } P \text{ and the } C_i, 2 ≤ i ≤ k, \text{ contains exactly one vertex in } \{v_1, \cdots, v_k\}, \text{ and } V(P) ∪ (\bigcup_{i=2}^{k} V(C_i)) ≥ N_L(H).
\end{equation}

**Proof.** Suppose that there exist vertex disjoint subgraphs $P, C_i, 2 ≤ i ≤ k$, in $L$ which satisfy (2.5). Write $V(P) ∩ \{v_1, \cdots, v_k\} = \{v_{j_1}\}$ and $V(C_i) ∩ \{v_1, \cdots, v_k\} = \{v_{j_i}\}$ for each $2 ≤ i ≤ k$. Let $u_r$ and $u_t$ be the initial vertex and the terminal vertex of $P$, respectively. Since $d_L(u) > l - s ≥ l - k + l - \frac{1}{2}(|V(P)| + \sum_{i=2}^{k} |V(C_i)|)$ for all $u ∈ V(L) - N_L(H)$ by Claim 4, it follows from Claim 1 that there are vertex disjoint subgraphs $P', C_i', 2 ≤ i ≤ k$, in $L$.
such that $P'$ is a path with $v_{ji} \in V(P')$ connecting $u_r$ and $u_t$, $C''_i$ is a cycle with $v_{ji} \in V(C''_i)$ for each $2 \leq i \leq k$, and $V(P') \cup (\bigcup_{i=2}^k V(C''_i)) = V(L)$. Take $h \in N_H(u_r)$ and $h' \in N_H(u_t) - \{h\}$ (note that $d_{H}(u_t) \geq 2$ because $u_t \in S$).

Combining $P'$ and a path in $H$ connecting $h$ and $h'$, we obtain a cycle $C'_i$. Then $C'_i$ and the $C''_i$, $2 \leq i \leq k$, are vertex disjoint cycles in $G$ such that $v_{ji} \in V(C'_i)$, $v_{ji} \in V(C''_i)$ for each $2 \leq i \leq k$, and $|V(C'_i)| + \sum_{i=2}^k |V(C''_i)| > \sum_{i=1}^k |V(C_i)|$. This contradicts the maximality of $\sum_{i=1}^k |V(C_i)|$.

For each $i \in \{1, \cdots, s\}$, take $w_i \in V(C_i) - \{u_i, v_i\}$ and let $W_i = \langle \{u_i, v_i, w_i\} \rangle_G$. We redefine the orientation of each $C_i$, $1 \leq i \leq s$, so that $w_i \in V(C_i(v_i, u_i))$.

**Claim 6.** Let $1 \leq h, j \leq s$ with $h \neq j$, and suppose that $v_h u_j \in E(G)$. Then $w_h v_j \notin E(G)$ or $w_h w_j \notin E(G)$.

**Proof.** At the cost of relabeling, we may assume $h = 1$ and $j = 2$. Suppose $v_1 u_2 \in E(G)$, $w_1 v_2 \in E(G)$ and $w_1 w_2 \in E(G)$. Let $P' = C_1[u_1, v_1]w_2$ and $C''_i = w_1 C'_2[w_2, w_1]v_1$, and let $C''_i = C_i$ for $3 \leq i \leq k$. Then $P'$ and the $C''_i (2 \leq i \leq k)$ satisfy (2.5), a contradiction.

Having (2.4) in mind, take $i_1$ and $i_2$ with $1 \leq i_1, i_2 \leq s$ and $i_1 \neq i_2$ satisfying $u_{i_2} \in N_G(v_{i_1})$ so that

\[
\sum_{i=1}^s d_{W_i}(w_{i_1}) \text{ is minimum.}
\]

We may assume $i_1 = 1$ and $i_2 = 2$. We remark in advance that we make use of (2.6) only in the last stage of the proof (see the paragraph following Claim 12).

**Claim 7.** We have $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) \leq 7$. Further if $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) = 7$, then

\[
\begin{cases}
  w_1 w_2 \notin E(G), \; w_1 u_1 \in E(G), \; w_1 v_1 \in E(G); \\
  v_2 x \in E(G) \text{ for each } x \in \langle \{u_1, v_1, w_1\} \rangle; \text{ and} \\
  w_2 x \in E(G) \text{ for each } x \in \langle u_1, v_1 \rangle.
\end{cases}
\]

**Proof.** Applying Claim 6 with $h = 1$ and $j = 2$, we get $v_2 w_1 \notin E(G)$ or $w_2 w_1 \notin E(G)$. This implies $d_{W_1}(v_2) + d_{W_1}(w_2) \leq 5$, and hence $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) \leq 7$. Now assume that $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) = 7$, i.e., $d_{W_1}(v_2) + d_{W_1}(w_2) = 5$ and $d_{W_1}(w_1) = 2$. Assume further that $w_1 w_2 \in E(G)$. Then $v_2 w_1 \notin E(G)$ by Claim 6. Hence by the assumption that $d_{W_1}(v_2) + d_{W_1}(w_2) = 5$, we in particular have $v_2 w_1 \in E(G)$ and $w_2 u_1 \in E(G)$. But then applying Claim 6 with $h = 2$ and $j = 1$, we get a contradiction. Thus the equality $d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) = 7$ implies (2.7).
Claim 8. \( d_{W_2}(w_1) + d_{W_2}(v_2) + d_{W_2}(w_2) \leq 6 \).

Proof. Since \( d_{W_2}(w_1) \leq 2 \) by Claim 6, the desired inequality follows. \( \square \)

Claim 9. \( d_{W_i}(w_1) + d_{W_i}(v_2) + d_{W_i}(w_2) \leq 7 \) for each \( i \in \{3, \ldots, s\} \).

Proof. Let \( 3 \leq i \leq s \). If \( v_2u_i \notin E(G) \), then \( d_{W_i}(v_2) \leq 2 \); if \( v_2u_i \in E(G) \), then \( d_{W_i}(v_2) \leq 2 \) by Claim 6. In either case, we have \( d_{W_i}(v_2) + d_{W_i}(w_2) \leq 5 \). Now assume that \( d_{W_i}(v_2) + d_{W_i}(w_2) = 5 \) and \( d_{W_i}(w_1) = 3 \). Then \( u_i \in N_G(v_2) \cap N_G(w_2) \) or \( u_i \in N_G(v_2) \cap N_G(w_2) \). If \( u_i \in N_G(v_2) \cap N_G(w_2) \), then \( P'_i = C_1[u_1, v_1]u_2, C'_2 = u_iC_2[v_2, w_2]u_i, C'_1 = w_1C_i[v_i, w_i]w_1 \) and \( C'_j = C_j, j \neq 1, 2, i \), satisfy (2.5), a contradiction; if \( u_i \in N_G(v_2) \cap N_G(w_2) \), then \( P'_i = C_1[u_1, v_1]u_2, C'_2 = w_1C_2[v_2, w_2]w_1, C'_i = w_1C_i[u_i, w_i]w_1 \) and \( C'_j = C_j, j \neq 1, 2, i \), satisfy (2.5), a contradiction. \( \square \)

Let \( W = \langle \bigcup_{i=1}^s V(W_i) \rangle_G \). Clearly

\[
\begin{aligned}
&d_L(x) \leq |V(L) - V(W)| + d_W(x) = l - 3s + d_W(x) \\
&\text{for each } x \in \{w_1, v_2, w_2\}.
\end{aligned}
\]

From this and Claims 7, 8 and 9, it follows that

\[
\begin{aligned}
d_L(w_1) + d_L(v_2) + d_L(w_2) \leq 3l - 9s + 7 + 6 + 7(s - 2) = 3l - 2s - 1.
\end{aligned}
\]

Now if \( d_L(w_1) + d_L(v_2) + d_L(w_2) \leq 3l - 2s - 2 \), then \( 3l - 2s - 2 \geq 3 \left( l - s + \frac{k - c_k}{3} + 1 \right) \) by Claim 4, and hence \( s \geq k - c_k + 5 \geq k + 1 \), a contradiction. Thus \( d_L(w_1) + d_L(v_2) + d_L(w_2) = 3l - 2s - 1 \). Then again by Claim 4, \( 3l - 2s - 1 \geq 3 \left( l - s + \frac{k - c_k}{3} + 1 \right) \), i.e., \( s \geq k + 4 - c_k \). This inequality holds only when \( c_k = 4 \) and \( s = k \). Thus \( s = k \equiv 1 \ (\text{mod} \ 3) \). Consequently, \( s \geq 4 \), and all of the following equalities, (2.10) through (2.12), must hold:

\[
\begin{aligned}
d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_1}(w_2) &= 7; \\
d_{W_2}(w_1) + d_{W_2}(v_2) + d_{W_2}(w_2) &= 6; \text{ and} \\
d_{W_i}(w_1) + d_{W_i}(v_2) + d_{W_i}(w_2) &= 7 \text{ for each } i \in \{3, \ldots, s\}.
\end{aligned}
\]

Note that we have not made use of (2.6) so far. Note also that we have

\[
\begin{aligned}
u_1 \in N_G(v_2)
\end{aligned}
\]
by (2.7) and (2.10). Thus \( d_{W_2}(w_2) + d_{W_2}(v_1) + d_{W_2}(w_1) = 7 \), and hence (2.7) holds with the roles of \( W_1 \) and \( W_2 \) replaced by each other. Consequently

\[
\begin{aligned}
w_1w_2 \notin E(G), w_1u_1 \in E(G), w_1v_1 \in E(G), \text{ and every vertex in } \{u_1, v_1, w_1\} \text{ and every vertex in } \{w_2, v_2, w_2\} \text{, except } w_1 \text{ and } w_2 \text{ and except possibly } u_1 \text{ and } u_2, \text{ are adjacent to each other.}
\end{aligned}
\]
Again note that we have not yet used (2.6). Thus
\[
(2.15) \quad \text{for any } i, j \in \{1, \cdots, s\} \text{ with } i \neq j \text{ such that } u_j \in N_G(v_i), \text{ the statements corresponding to (2.10) through (2.12) and (2.14) hold.}
\]

Let \( I = \{ i \mid d_{W_i}(w_1) = 3, 3 \leq i \leq s \} \) and let \( J = \{3, \cdots, s\} - I \). Since \( d_{W}(w_1) \geq 2s + 1 \) by Claim 4 and (2.8), and since \( d_{W_1}(w_1) = d_{W_2}(w_1) = 2 \) by (2.14), we have
\[
(2.16) \quad I \neq \emptyset.
\]

**Claim 10.** Let \( i \in I \). Then \( w_iv_1, w_iv_2 \notin E(G) \) and \( d_{W_i}(w_2) = 3 \).

**Proof.** We first show that \( u_iv_2 \notin E(G) \). Suppose that \( u_iv_2 \in E(G) \). Since \( d_{W_1}(w_1) = 3 \) by assumption, and since \( d_{W_1}(v_2) + d_{W_2}(w_2) = 5 \) by (2.14) and (2.15), we have \( d_{W_1}(w_1) + d_{W_1}(v_2) + d_{W_2}(w_2) = 8 \), which contradicts (2.12). Thus \( u_iv_2 \notin E(G) \). Next suppose that \( w_iv_2 \in E(G) \). If \( u_iv_2, v_jw_2 \in E(G) \), then \( P' = C_1[u_1, v_1]u_2, C'_2 = w_1v_2w_iv_1, C'_3 = w_2C_1[u_1, v_1]w_2 \) and \( C'_j = C_j, j \neq 1, 2, i, \) satisfy (2.5), a contradiction; otherwise, in view of (2.12), \( w_iv_2 \in E(G) \), and hence \( P' = C_1[u_1, v_1]u_2, C'_2 = w_1C_2[v_2, w_2]w_i, C'_3 = w_1C_1[u_1, v_1]w_1 \) and \( C'_j = C_j, j \neq 1, 2, i, \) satisfy (2.5), a contradiction. Thus \( w_iv_2 \notin E(G) \). Consequently \( d_{W_i}(v_2) = 1 \) and
\[
(2.17) \quad d_{W_i}(w_2) = 3
\]
by (2.12). Now by (2.13) and (2.17), we can argue as above with the roles of \( W_1 \) and \( W_2 \) replaced by each other, to obtain \( (u_iv_1 \notin E(G) \) and \( w_jw_1 \notin E(G) \). \( \square \)

**Claim 11.** Let \( j \in J \). Then \( w_jw_1 \notin E(G) \), and \( u_jv_1, u_jw_1, v_jw_1 \in E(G) \).

**Proof.** We have
\[
(2.18) \quad d_{W_j}(w_1) + d_{W_j}(v_2) + d_{W_j}(w_2) = 7
\]
by (2.12), and \( d_{W_j}(w_1) \leq 2 \) by the assumption that \( j \in J \). Furthermore, it follows from (2.12), (2.13) and (2.15) that
\[
(2.19) \quad d_{W_j}(w_2) + d_{W_j}(v_1) + d_{W_j}(w_1) = 7.
\]
If \( d_{W_j}(w_2) = 3 \), then applying Claim 10 with the roles of \( W_1 \) and \( W_2 \) replaced by each other, we obtain \( d_{W_j}(w_1) = 3 \), which contradicts the assumption that \( j \in J \). Thus \( d_{W_j}(w_2) \leq 2 \). Consequently by (2.18) and (2.19), \( d_{W_j}(v_1) = d_{W_j}(v_2) = 3 \). Therefore \( u_jv_1 \in E(G) \), and hence \( w_jw_1 \notin E(G) \) and \( u_jw_1, v_jw_1 \in E(G) \) by (2.14) and (2.15). \( \square \)
Since $v_1u_j \in E(G)$ for each $j \in J$ by Claim 11, we can apply Claim 10 to $W_1$ and $W_j$ in place of $W_1$ and $W_2$, to obtain the following claim:

**Claim 12.** Let $i \in I$ and let $j \in J$. Then $w_i v_j \not\in E(G)$. \hfill \□

Take $i' \in I$ (recall (2.16)). Since $\sum_{i=1}^{s} d_{W_i}(w_{i'}) \leq 2|J \cup \{1, 2, i'\}| + 3|I - \{i'\}| = 3|I| + 2|J| + 3$ by Claims 10 and 12, it follows from the choice of $i_1 = 1$ and $i_2 = 2$ (recall (2.6)) that

\[
(2.20) \quad \sum_{i=1}^{s} d_{W_i}(w_1) \leq 3|I| + 2|J| + 3.
\]

On the other hand, it follows from (2.14) and Claim 11 that

\[
\sum_{i=1}^{s} d_{W_i}(w_1) = |\{u_1,v_1,u_2,v_2\}| + 3|I| + 2|J| = 3|I| + 2|J| + 4,
\]

which contradicts (2.20).

This completes the proof of Theorem 1.

**References**


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